18.01A Recitation Partial Solutions — Monday, Sept. 10, 2018

Practice problems:

1. Find the quadratic approximation of $\cos(5x)$ at $x = 0$

   (a) by using the general formula $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.

   (b) by using the quadratic approximation $\cos x \approx 1 - \frac{x^2}{2}$.

   Compare the two results.

2. Find the quadratic approximation of

$$\frac{1}{(1-2x)(1-3x)}$$

   at $x = 0$ by using the basic approximation formulas.

3. Find the quadratic approximation of

$$\frac{(1+x)^{\frac{3}{2}}}{1+2x}$$

   at $x = 0$ by using the basic formulas.

4. Evaluate the following limits.

   (a)

$$\lim_{x \to -\infty} xe^x$$

   (b)

$$\lim_{x \to 0} x^{x^2}$$

   (c)

$$\lim_{x \to 0} \frac{\sin 2x - 2\sin x}{\sin 3x - 3\sin x}$$

   Solution.

$$\begin{aligned}\lim_{x \to 0} \frac{\sin 2x - 2\sin x}{\sin 3x - 3\sin x} &= \lim_{x \to 0} \frac{2\cos 2x - 2\cos x}{3\cos 3x - 3\cos x} = \lim_{x \to 0} \frac{-4\sin 2x + 2\sin x}{-9\sin 3x + 3\sin x} \\ &= \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{-27\cos 3x + 3\cos x} = -\frac{1}{4}.\end{aligned}$$

   □

   Remark: when you use L'Hôpital's rule, please make sure that 1. you really need to use it, i.e., do you really have an indeterminant form; 2. you can use it, i.e., does it really satisfy the assumption of L'Hôpital's rule.

(d)

$$\lim_{x\to\frac{\pi}{4}} \frac{\ln(\tan x)}{\sin x - \cos x}$$

Solution.

$$\lim_{x\to\frac{\pi}{4}} \frac{\ln(\tan x)}{\sin x - \cos x} = \lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x}\frac{\ln(\tan x)}{\tan x - 1} = \left(\lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x}\right)\cdot\left(\lim_{x\to\frac{\pi}{4}} \frac{\ln(\tan x)}{\tan x - 1}\right)$$
$$= \sqrt{2}\lim_{u\to 1}\frac{\ln u}{u-1} = \sqrt{2}\lim_{u\to 1}\frac{\frac{1}{u}}{1} = \sqrt{2}.$$

□

(e)

$$\lim_{x\to 0}\frac{e^{2x}-1}{\sin 5x}$$

Solution.

$$\lim_{x\to 0}\frac{e^{2x}-1}{\sin 5x} = \lim_{x\to 0}\frac{2e^{2x}}{5\cos(5x)} = \frac{2}{5}.$$

□

(f)

$$\lim_{x\to\infty}\frac{\ln(\ln x)}{\ln x}$$

Solution.

$$\lim_{x\to\infty}\frac{\ln(\ln x)}{\ln x} = \lim_{u\to\infty}\frac{\ln u}{u} = \lim_{u\to\infty}\frac{\frac{1}{u}}{1} = 0.$$

□

(g)

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x}}$$

Solution.

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(\cos x)} = e^{\lim_{x\to 0}\frac{1}{x}\ln(\cos x)} = e^{\lim_{x\to 0}\frac{-\tan x}{1}} = e^0 = 1.$$

□

(h)

$$\lim_{x\to\infty} e^{-x}\ln x$$

Solution.

$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = 0.$$

□

(i) (This is a hard one.)

$$\lim_{x\to 0} \cot x - \frac{1}{x}$$

Solution.

$$\lim_{x\to 0} \cot x - \frac{1}{x} = \lim_{x\to 0} \frac{\cos x}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{-x\sin x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = 0.$$

□