18.01A Recitation Partial Solution — Wednesday, Sept. 12, 2018

Practice problems:

1. Find $\int_0^1 e^x dx$ by using Riemann sum.
2. Find the limit by relating it to Riemann sum:

$$\lim_{n\to\infty} \frac{\cos(1/n) + \cos(2/n) + \cdots + \cos((n-1)/n) + \cos(1)}{n}$$

3. In class, we saw that for continuous functions, the limit of the Riemann sum does not depend on the choice of c_i inside each interval. For non-continuous functions, the story can **sometimes** be different. Consider the function $h(x)$ defined on $[0, 1]$:

$$h(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that one cannot define $\int_0^1 h(x)dx$ by showing that the limit of the upper sum and the limit of the lower sum are different.

Solution. Recall that the upper (lower resp.) sum is the Riemann sum that one chooses $c_i \in [x_i, x_{i+1}]$ such that $h(c_i)$ is the maximum (minimum resp.) of h over $[x_i, x_{i+1}]$.

Notice that on each subinterval, the maximum of the function h is always 1 while the minimum of the function h is always 0. (Why?) This implies that the upper sum is always 1 (regardless of what n is) and the lower sum is always 0 (regardless of what n is). Hence, when we take the limit as $n \to \infty$, the limit of the upper sum is 1 and the limit of the lower sum is 0. So, for this particular function h, different Riemann sums can converge to different numbers. When such things happen, we say that the function h is not integrable.

Fortunately, for continuous functions, this will NEVER happen. (As a side question, can you see that h is discontinuous everywhere?) $\square$