18.01A Recitation Solutions — Monday, Sept. 24, 2018

Practice problems:

1.
$$\int \sin^3(x)dx$$

Solution.
$$\int \sin^3(x)dx = \int \sin^2(x)\cdot\sin(x)dx.$$
Make the substitution $u=\cos(x)$. Then
$$\int \sin^2(x)\cdot\sin(x)dx = -\int(1-u^2)du = -u+\frac{1}{3}u^3+C = -\cos x+\frac{1}{3}\cos^3 x+C.$$
□

2.
$$\int \frac{dx}{x\sqrt{a^2-x^2}}$$

Solution. Make the substitution $x=a\sin\theta$. We have
$$\int \frac{dx}{x\sqrt{a^2-x^2}} = \int \frac{a\cos\theta d\theta}{a\sin\theta a\cos\theta} = \int \frac{1}{a\sin\theta}d\theta = \frac{1}{a}\int \frac{1}{\sin\theta}d\theta.$$
Let us now compute $\int \frac{1}{\sin\theta}d\theta$. Make the substitution $\theta = 2t$
$$\begin{aligned}
\int \frac{1}{\sin\theta}d\theta &= \int \frac{1}{\sin 2t}2dt = \int \frac{1}{2\sin t\cos t}2dt = \int \frac{1}{\sin t\cos t}dt \\
&= \int \frac{\sin^2 t+\cos^2 t}{\sin t\cos t}dt = \int \frac{\sin^2 t}{\sin t\cos t}dt + \int \frac{\cos^2 t}{\sin t\cos t}dt \\
&= \int \frac{\sin t}{\cos t}dt + \int \frac{\cos t}{\sin t}dt \\
&= -\ln|\cos t| + \ln|\sin t| + C \\
&= \ln|\tan t| + C \\
&= \ln|\tan\frac{\theta}{2}| + C.
\end{aligned}$$

Since the original integral is in terms of x, we need to express $\tan\frac{\theta}{2}$ in terms of x. Notice that we know $\sin\theta = \frac{x}{a}$.
$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \frac{\frac{1}{2}\sin\theta}{\frac{1}{2}(\cos\theta+1)} = \frac{\sin\theta}{\cos\theta+1} = \frac{\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}+1} = \frac{x}{\sqrt{a^2-x^2}+a}.$$

Hence,
$$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln|\frac{x}{\sqrt{a^2-x^2}+a}|+C.$$
□

3.

$$\int_1^{\sqrt{2}} \frac{dx}{x^3\sqrt{x^2-1}}$$

Solution. Make the substitution $x = \sec\theta$.

$$\int_1^{\sqrt{2}} \frac{dx}{x^3\sqrt{x^2-1}} = \int_0^{\pi/4} \frac{\sec\theta\tan\theta}{\sec^3\theta|\tan\theta|}d\theta.$$

Notice that like we talked about in class, when you are doing definite integrals, you do care about the absolute value! In this case, however, $\tan\theta$ when $0 < \theta < \pi/4$ is always positive. Hence, we may lose the absolute value sign.

$$\begin{aligned}\int_0^{\pi/4} \frac{\sec\theta\tan\theta}{\sec^3\theta|\tan\theta|}d\theta &= \int_0^{\pi/4} \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}d\theta = \int_0^{\pi/4} \frac{1}{\sec^2\theta}d\theta \\ &= \int_0^{\pi/4} \cos^2\theta d\theta = \int_0^{\pi/4} \frac{1}{2}(1+\cos 2\theta)d\theta = \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}.\end{aligned}$$

□

4.

$$\int_{-\sqrt{2}}^{-1} \frac{dx}{x^3\sqrt{x^2-1}}$$

Solution. You need to compare this question with the previous one.

Make the substitution $x = \sec\theta$.

$$\int_{-\sqrt{2}}^{-1} \frac{dx}{x^3\sqrt{x^2-1}} = \int_{\frac{3}{4}\pi}^{\pi} \frac{\sec\theta\tan\theta}{\sec^3\theta|\tan\theta|}d\theta.$$

Note that when $\frac{3}{4}\pi < \theta < \pi$, $\tan\theta$ is negative. Hence, $|\tan\theta| = -\tan\theta$.

$$\begin{aligned}\int_{\frac{3}{4}\pi}^{\pi} \frac{\sec\theta\tan\theta}{\sec^3\theta|\tan\theta|}d\theta &= -\int_{\frac{3}{4}\pi}^{\pi} \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}d\theta = -\int_{\frac{3}{4}\pi}^{\pi} \frac{1}{\sec^2\theta}d\theta \\ &= -\int_{\frac{3}{4}\pi}^{\pi} \cos^2\theta d\theta = -\int_{\frac{3}{4}\pi}^{\pi} \frac{1}{2}(1+\cos 2\theta)d\theta = -\left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_{\frac{3}{4}\pi}^{\pi} = -\frac{\pi}{8} - \frac{1}{4}.\end{aligned}$$

□

5.

$$\int \sqrt{a^2+x^2}dx$$

(Hint: $\int \frac{1}{\cos^3\theta}d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$.)

Solution. Make the substitution $x = a\tan\theta$.

$$\int \sqrt{a^2+x^2}dx = \int a\sec\theta a\sec^2\theta d\theta = a^2\int \sec^3\theta d\theta.$$

Using the hint, we have

$$\int \sqrt{a^2+x^2}dx = \frac{a^2}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.$$

Now we need to write our final answer in terms of x. Notice that $\tan\theta = \frac{x}{a}$. Hence $\sec\theta = \sqrt{1+\frac{x^2}{a^2}}$. Thus,

$$\int \sqrt{a^2+x^2}dx = \frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}\ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|+C = \frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}\ln\left|x+\sqrt{a^2+x^2}\right|+C.$$

□

6. (If time permits.) If the circle $(x-b)^2+y^2 = a^2(0 < a < b)$ is revolved about the y-axis, the resulting solid of revolution is called a *torus*. Find the volume of this torus.

   Solution. We will only set up the integral. If we use the shell method (partition the x-axis), the volume element is given by

   $$dV = 2\pi x 2y dx = 4\pi x\sqrt{a^2-(x-b)^2}dx.$$

   Hence, the integral is:

   $$\int_{b-a}^{b+a} 4\pi x\sqrt{a^2-(x-b)^2}dx.$$

   It's a good exercise to try and evaluate this integral. □