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THE L_p ALEKSANDROV PROBLEM FOR ORIGIN-SYMMETRIC POLYTOPES

YIMING ZHAO

Department of Mathematics, Massachusetts Institute of Technology

Abstract. The L_p Aleksandrov integral curvature and its corresponding characterization problem—the L_p Aleksandrov problem—were recently introduced by Huang, Lutwak, Yang, and Zhang. The current work presents a solution to the L_p Aleksandrov problem for origin-symmetric polytopes when $-1 < p < 0$.

1. Introduction

The classical Aleksandrov problem is the counterpart of the Minkowski problem—a fundamental problem in the Brunn-Minkowski theory whose influence reaches many fields of mathematics including convex geometry, differential geometry, PDE, and functional analysis. The Aleksandrov problem is the measure characterization problem for Aleksandrov integral curvature $J(K, \cdot)$ (also known as integral Gauss curvature), the most studied curvature measure which was defined by Aleksandrov [1]. When the convex body K is sufficiently smooth, the Aleksandrov integral curvature of K (when viewed as a measure on ∂K) has the Gauss curvature as its density.

The Aleksandrov problem was completely solved by Aleksandrov himself using a topological argument, see Aleksandrov [1]. Alternative approaches that connect the Aleksandrov problem to optimal mass transport were given by Oliker [42] and more recently by Bertrand [4].

The last three decades saw the rapid and flourish development of the L_p Brunn-Minkowski theory that was initiated by Firey but only truly gained life when Lutwak [34, 35] began to systematically work on it in the early 1990s. The L_p Brunn-Minkowski theory is arguably the most vibrant theory in modern convex geometry and has been the breeding ground for many important results. The L_p Minkowski problem is the fundamental problem in the L_p Brunn-Minkowski theory and characterizes the L_p surface area measure that sits in the center of the theory. In particular, the discovery of an important class of affine isoperimetric inequalities—the sharp affine L_p Sobolev inequality—owes to the solution of the L_p Minkowski problem for $p \geq 1$, see [38]. This effort has over the years inspired many more sharp affine isoperimetric inequalities, see [20, 37, 38].

E-mail address: `yimingzh@mit.edu`.

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The corresponding measure characterization problem (geometric measure, resp.) for the Aleksandrov problem (Aleksandrov integral curvature, resp.) in the L_p Brunn-Minkowski theory had been long sought-for. In a recent groundbreaking work [24], Huang, Lutwak, Yang & Zhang (Huang-LYZ) discovered that Aleksandrov integral curvature naturally arises as the "differential" of a certain entropy integral. Following their work, they introduced the L_p Aleksandrov integral curvature in [25] and posed the measure characterization problem called the L_p Aleksandrov problem. More details will follow shortly.

The purpose of the current work is to solve the L_p Aleksandrov problem when $-1 < p < 0$ in the case of origin-symmetric polytopes.

We shall provide some background on the L_p Brunn-Minkowski theory.

The L_p surface area measure $S_p(K, \cdot)$ is the fundamental geometric measure in the L_p Brunn-Minkowski theory. In fact, many key concepts in the L_p theory including the L_p mixed volume and the L_p affine surface area can be defined solely using the L_p surface area measure.

The L_p Minkowski problem asks: given a Borel measure μ on S^{n-1}, what are necessary and sufficient conditions on μ so that there exists a convex body K such that μ is exactly the L_p surface area measure of K? When $p = 1$, the L_p Minkowski problem is the same as the classical Minkowski problem which was solved by Minkowski, Fenchel & Jessen, Aleksandrov, etc. Regularity results on the Minkowski problem include the influential paper [17] by Cheng & Yau. The solution, when $p > 1$, was given by Lutwak [34] when μ is an even measure and Chou & Wang [18] for arbitrary μ. See also Chen [16], Chen, Li & Zhu [14], Huang, Liu & Xu [23], Hug-LYZ [27], Jian, Lu & Wang [28], Lutwak & Oliker [36], LYZ [39], and Zhu [51].

The L_p Minkowski problem contains two major unsolved cases.

When $p = -n$, the L_{-n} surface area measure $S_{-n}(K, \cdot)$ is also known as the centro-affine surface area measure whose density in the smooth case is the centro-affine Gauss curvature. The characterization problem, in this case, is the centro-affine Minkowski problem posed in Chou & Wang [18]. See also Jian, Lu & Zhu [29], Lu & Wang [31], Zhu [50], etc., on this problem.

When $p = 0$, the L_0 surface area measure $S_0(K, \cdot)$ is the cone volume measure whose total measure is the volume of K. See, for example, [7, 9, 21, 32, 33, 41, 43, 45, 46, 49]. The characterization problem for the cone volume measure is the logarithmic Minkowski problem. A complete solution to the existence part of the logarithmic Minkowski problem, when restricting to even measures and the class of origin-symmetric convex bodies, was given by Böröczky-LYZ [10]. In the general case (non-even case), different efforts have been made by Böröczky, Hegedűs & Zhu [6], Stancu [45], Zhu [49], and most recently by Chen, Li & Zhu [15]. The logarithmic Minkowski problem has strong connections with isotropic measures (Böröczky-LYZ [11]) and curvature flows (Andrews [2, 3]).

In a groundbreaking work [24], Huang-LYZ discovered a new family of geometric measures called dual curvature measures $\widetilde{C}_q(K, \cdot)$ and the variational formula that leads to them. The dual Minkowski problem—the problem of prescribing dual curvature measures—was posed as well. The dual Minkowski problem miraculously contains problems such as the Aleksandrov problem ($q = 0$) and the logarithmic Minkowski problem ($q = n$) as special cases. The problem quickly became the center of attention, see, for example, [5, 8, 12, 13, 19, 22, 25, 26, 30, 40, 47, 48].

The variational formula for Aleksandrov's integral curvature obtained in [24] allowed the following discovery, see [25]: for each $0 \neq p \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$, define the L_p Aleksandrov integral curvature, $J_p(K, \cdot)$, of K as the unique Borel measure on S^{n-1} such that

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_p t \cdot Q)\bigg|_{t=0} = \frac{1}{p}\int_{S^{n-1}} \rho_Q(u)^{-p} dJ_p(K, u)$$

holds for every $Q \in \mathcal{K}_0^n$, where $\mathcal{E}(\cdot)$ is the entropy integral defined by

$$\mathcal{E}(Q) = -\int_{S^{n-1}} \log h_Q(v) dv, \tag{1.1}$$

and $K \hat{+}_p t \cdot Q$ is the harmonic L_p-combination defined by

$$K \hat{+}_p t \cdot Q = (K^* +_p t \cdot Q^*)^*.$$

Here K^* is the polar body of K.

The L_p Aleksandrov integral curvature is absolutely continuous with respect to the classical Aleksandrov integral curvature $J(K, \cdot)$:

$$dJ_p(K, \cdot) = \rho_K^p dJ(K, \cdot).$$

Hence $J_p(K, \cdot)$ is defined for $p = 0$ and $J_0(K, \cdot) = J(K, \cdot)$.

The L_p Aleksandrov problem is the measure characterization problem for the L_p Aleksandrov integral curvature.

Problem (The L_p Aleksandrov problem)**.** *Given a non-zero finite Borel measure μ on S^{n-1} and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on μ so that there exists $K \in \mathcal{K}_o^n$ such that $\mu = J_p(K, \cdot)$?*

When the given measure μ has a density f, solving the L_p Aleksandrov problem is the same as solving the following Monge-Ampère type equation on S^{n-1}:

$$h^{1-p}(|\nabla h|^2 + h^2)^{-\frac{n}{2}} \det(\nabla^2 h + hI) = f,$$

where h is the unknown, ∇h and $\nabla^2 h$ are the gradient and Hessian of h on S^{n-1} with respect to the standard metric, and I is the identity matrix.

Huang-LYZ established the existence part of the problem in several situations. When $p > 0$, the existence part is completely established.

Theorem 1.1 ([25])**.** *Suppose $p \in (0, \infty)$ and μ is a non-zero finite Borel measure on S^{n-1}. There exists $K \in \mathcal{K}_o^n$ such that $\mu = J_p(K, \cdot)$ if and only if μ is not concentrated in any closed hemisphere.*

The case $p < 0$ is much more complicated. Under very strong assumptions, the following existence result was established.

Theorem 1.2 ([25])**.** *Suppose $p \in (-\infty, 0)$ and μ is a non-zero even finite Borel measure on S^{n-1}. If μ vanishes on all great subspheres of S^{n-1}, then there exists $K \in \mathcal{K}_o^n$ such that $\mu = J_p(K, \cdot)$.*

Note that the conditions in Theorem 1.2 are quite strong. In particular, an important class of convex bodies—polytopes—are not included in the solution, for the simple reason that the L_p Aleksandrov integral curvature of a polytope must be discrete and therefore must obtain positive measure on many great subspheres. In fact, the classical Minkowski problem and some cases of the L_p Minkowski problem were first solved for the polytopal case and then solved for the general case using approximation.

Theorem 1.2 was shown using variational method. What makes the problem especially challenging in the case when μ has a positive concentration in one of the proper subspaces is the behavior of the functional Φ in the maximization problem (3.1). See also (3.2). When the given measure μ has even the slightest concentration in a proper subspace $u^\perp$, the functional Φ will still obtain a finite number for any convex body in $u^\perp$. This feature of Φ made it extremely challenging to show that the final solution K does not collapse into a lower dimensional subspace.

The aim of the current work is to show that the L_p Aleksandrov problem has a solution when $-1 < p < 0$ and the given measure μ is an even discrete measure.

Theorem 1.3. *Suppose $p \in (-1, 0)$ and μ is a non-zero, even, discrete, finite, Borel measure on S^{n-1}. There exists an origin-symmetric polytope $K \in \mathcal{K}_e^n$ such that $\mu = J_p(K, \cdot)$ if and only if μ is not concentrated entirely on any great subspheres.*

Note that although still variational in nature, the approach here is vastly different from that in [25]. It should also be pointed out that there might be a way to use the solution obtained in the current work to obtain a solution to the even L_p Aleksandrov problem for $-1 < p < 0$ via approximation.

2. Preliminaries

This section is divided into two subsections. In the first subsection, basics in the theory of convex bodies will be covered. In the second subsection, the notion of L_p Aleksandrov integral curvature and L_p Aleksandrov problem will be introduced.

2.1. Basics in the theory of convex bodies.

The book [44] by Schneider offers a comprehensive overview of the theory of convex bodies.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1} and the volume of the unit ball will be written as ω_n. A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. The boundary of K is written as ∂K. Denote by $\mathcal{K}_0^n$ the class of convex bodies that contain the origin in their interiors in $\mathbb{R}^n$ and by $\mathcal{K}_e^n$ the class of origin-symmetric convex bodies in $\mathbb{R}^n$.

Let K be a compact convex subset of $\mathbb{R}^n$. The support function h_K of K is defined by

$$h_K(y) = \max\{x \cdot y : x \in K\}, \quad y \in \mathbb{R}^n.$$

The support function h_K is a continuous function homogeneous of degree 1. Suppose K contains the origin in its interior. The radial function ρ_K is defined by

$$\rho_K(x) = \max\{\lambda : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The radial function ρ_K is a continuous function homogeneous of degree -1. It is not hard to see that $\rho_K(u)u \in \partial K$ for all $u \in S^{n-1}$.

For a convex body $K \in \mathcal{K}_0^n$, the polar body of K is given by

$$K^* = \{y \in \mathbb{R}^n : y \cdot x \leq 1, \text{ for all } x \in K\}.$$

It is simple to check that $K^* \in \mathcal{K}_0^n$ and that

$$h_{K^*}(x) = 1/\rho_K(x) \qquad \text{and} \qquad \rho_{K^*}(x) = 1/h_K(x),$$

for $x \in \mathbb{R}^n \setminus \{o\}$. Moreover, we have $(K^*)^* = K$.

For each $f \in C^+(S^{n-1})$, the Wulff shape $[f]$ generated by f is the convex body defined by

$$[f] = \{x \in \mathbb{R}^n : x \cdot v \leq f(v), \text{ for all } v \in S^{n-1}\}.$$

It is apparent that $h_{[f]} \leq f$ and $[h_K] = K$ for each $K \in \mathcal{K}_0^n$.

The L_p combination of two convex bodies $K, L \in \mathcal{K}_0^n$ was first studied by Firey and was the starting point of the now rich L_p Brunn-Minkowski theory developed by Lutwak [34, 35]. For $t, s > 0$, the L_p combination of K and L, denoted by $t \cdot K +_p s \cdot L$, is defined to be the Wulff shape generated by the function $h_{t,s}$ where

$$h_{t,s} = \begin{cases} (th_K^p + sh_L^p)^{\frac{1}{p}}, & \text{if } p \neq 0, \\ h_K^t h_L^s, & \text{if } p = 0. \end{cases}$$

When $p \geq 1$, by the convexity of ℓ_p norm, we get that

$$h^p_{K +_p t \cdot L} = h_K^p + th_L^p.$$

Define the L_p harmonic combination $t \cdot K \hat{+}_p s \cdot L$ by

$$t \cdot K \hat{+}_p s \cdot L = (t \cdot K^* +_p s \cdot L^*)^*.$$

Suppose K_i is a sequence of convex bodies in $\mathbb{R}^n$. We say K_i converges to a compact convex subset $K \subset \mathbb{R}^n$ if

$$\max\{|h_{K_i}(v) - h_K(v)| : v \in S^{n-1}\} \to 0, \tag{2.1}$$

as $i \to \infty$. If K contains the origin in its interior, equation (2.1) implies

$$\max\{|\rho_{K_i}(u) - \rho_K(u)| : u \in S^{n-1}\} \to 0,$$

as $i \to \infty$.

For a compact convex subset K in $\mathbb{R}^n$ and $v \in S^{n-1}$, the supporting hyperplane $H(K, v)$ of K at v is given by

$$H(K, v) = \{x \in K : x \cdot v = h_K(v)\}.$$

By its definition, the supporting hyperplane $H(K, v)$ is non-empty and contains only boundary points of K. For $x \in H(K, v)$, we say v is an outer unit normal of K at $x \in \partial K$.

Let $\omega \subset S^{n-1}$ be a Borel set. The radial Gauss image of K at ω, denoted by $\boldsymbol{\alpha}_K(\omega)$, is defined to be the set of all outer unit normals v of K at some boundary point $u\rho_K(u)$ where $u \in \omega$, i.e.,

$$\boldsymbol{\alpha}_K(\omega) = \{v \in S^{n-1} : v \cdot u\rho_K(u) = h_K(v) \text{ for some } u \in \omega\}.$$

When $\omega = \{u\}$ is a singleton, we usually write $\boldsymbol{\alpha}_K(u)$ instead of the more cumbersome notation $\boldsymbol{\alpha}_K(\{u\})$. Let ω_K be the subset of S^{n-1} such that $\boldsymbol{\alpha}_K(u)$ contains more than one element for each $u \in \omega_K$. By Theorem 2.2.5 in [44], the set ω_K has spherical Lebesgue measure 0. The radial Gauss map of K, denoted by α_K, is the map defined on $S^{n-1} \setminus \omega_K$ that takes each point u in its domain to the unique vector in $\boldsymbol{\alpha}_K(u)$. Hence α_K is defined almost everywhere on S^{n-1} with respect to the spherical Lebesgue measure.

Let $\eta \subset S^{n-1}$ be a Borel set. The reverse radial Gauss image of K, denoted by $\boldsymbol{\alpha}_K^*(\eta)$, is defined to be the set of all radial directions such that the corresponding boundary points have at least one outer unit normal in η, i.e.,

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in S^{n-1} : v \cdot u\rho_K(u) = h_K(v) \text{ for some } v \in \eta\}.$$

When $\eta = \{v\}$ is a singleton, we usually write $\boldsymbol{\alpha}^*_K(v)$ instead of the more cumbersome notation $\boldsymbol{\alpha}^*_K(\{v\})$. Let η_K be the subset of S^{n-1} such that $\boldsymbol{\alpha}^*_K(v)$ contains more than one element for each $v \in \eta_K$. By Theorem 2.2.11 in [44], the set η_K has spherical Lebesgue measure 0. The reverse radial Gauss map of K, denoted by α^*_K, is the map defined on $S^{n-1} \setminus \eta_K$ that takes each point v in its domain to the unique vector in $\boldsymbol{\alpha}^*_K(v)$. Hence α^*_K is defined almost everywhere on S^{n-1} with respect to the spherical Lebesgue measure.

2.2. L_p **Aleksandrov integral curvature and the** L_p **Aleksandrov problem.** For $K \in \mathcal{K}^n_0$, the Aleksandrov integral curvature of K, denoted by $J(K, \cdot)$, is a Borel measure on S^{n-1} given by

$$J(K, \omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega)). \tag{2.2}$$

It is apparent that $J(K, S^{n-1}) = n\omega_n$. The classical Aleksandrov problem is the measure characterization problem for Aleksandrov integral curvature: *given a Borel measure μ on S^{n-1} with $|\mu| = n\omega_n$, under what conditions on μ is there a convex body $K \in \mathcal{K}^n_0$ such that $\mu = J(K, \cdot)$?*

The Aleksandrov problem was completely solved by Aleksandrov himself [1] using his mapping lemma. In particular, there exists a $K \in \mathcal{K}^n_0$ with $\mu = J(K, \cdot)$ if and only if the given measure μ satisfies the following Aleksandrov condition:

$$\mu(\omega) < \mathcal{H}^{n-1}(S^{n-1} \setminus \omega^*),$$

for each non-empty spherically convex $\omega \subset S^{n-1}$. Here ω^* is given by

$$\omega^* = \{v \in S^{n-1} : v \cdot u \le 0, \forall u \in \omega\}.$$

Moreover, the convex body K, if it exists, is unique up to a dilation.

Aleksandrov integral curvature arises naturally by "differentiating" the entropy integral $\mathcal{E}$ given in (1.1), see [25]. In particular,

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_o t \cdot Q)\bigg|_{t=0} = -\int_{S^{n-1}} \log \rho_Q(u) dJ(K, u) \tag{2.3}$$

holds for each $Q \in \mathcal{K}^n_0$. Note that, instead of defining Aleksandrov integral curvature as in (2.2), one may define $J(K, \cdot)$ as the unique Borel measure on S^{n-1} such that (2.3) holds for each $Q \in \mathcal{K}^n_0$. This motivates the discovery of the L_p Aleksandrov integral curvature [25]. For $K \in \mathcal{K}^n_0$ and $p \neq 0$, the L_p Aleksandrov integral curvature of K, $J_p(K, \cdot)$, is defined to be the unique Borel measure on S^{n-1} such that

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_p t \cdot Q)\bigg|_{t=0} = \frac{1}{p}\int_{S^{n-1}} \rho_Q(u)^{-p} dJ_p(K, u)$$

holds for each $Q \in \mathcal{K}^n_0$.

The L_p Aleksandrov integral curvature has the following integral representation:

$$dJ_p(K, \cdot) = \rho_K^p dJ(K, \cdot).$$

Hence, we may define $J_0(K, \cdot)$ as the classical Aleksandrov integral curvature $J(K, \cdot)$.

When the body K is sufficiently smooth, the L_p Aleksandrov integral curvature $J(K, \cdot)$ is absolute continuous with respect to the spherical Lebesgue measure and its Radon-Nikodym derivative is given by

$$h^{1-p}(|\nabla h|^2 + h^2)^{-\frac{n}{2}} \det(\nabla^2 h + hI), \tag{2.4}$$

where ∇h and $\nabla^2 h$ are the gradient and Hessian of h on S^{n-1} with respect to an orthonormal basis.

Huang-LYZ [25] posed the following L_p Aleksandrov problem.

Problem (The L_p Aleksandrov problem)**.** *Given a non-zero finite Borel measure μ on S^{n-1} and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on μ so that there exists $K \in \mathcal{K}_o^n$ such that $\mu = J_p(K, \cdot)$?*

By (2.4), the L_p Aleksandrov problem reduces to the following PDE when the given measure μ has a density f:

$$h^{1-p}(|\nabla h|^2 + h^2)^{-\frac{n}{2}} \det(\nabla^2 h + hI) = f.$$

When $p = 0$, the L_0 Aleksandrov problem is nothing but the classical Aleksandrov problem. When $p > 0$, the existence part of the L_p Aleksandrov problem was completely settled in Huang-LYZ [25], see Theorem 1.1. However, when $p < 0$, a relatively strong condition was required in Huang-LYZ [25] to show the existence, see Theorem 1.2. This condition excludes a very important subclass of convex bodies—polytopes. It is the aim of the current work to fill that gap in the case when $-1 < p < 0$ and the polytope is origin-symmetric.

The proof adopted here is variational in nature. In Section 3, we shall convert the L_p Aleksandrov problem, when the given measure is discrete and even, to an optimization problem. In Section 4, the proposed optimization problem will be solved. The proof to the main theorem, Theorem 1.3, is given at the end of Section 4.

3. Optimization problem

The following lemma was given in Huang-LYZ [25], which connects the L_p Aleksandrov problem to an optimization problem.

Lemma 3.1 (Lemma 5.3, [25])**.** *Suppose $p \neq 0$. Let μ be a finite even, Borel measure on S^{n-1} and $K \in \mathcal{K}_e^n$ be a body such that*

$$\Psi(K) = \sup\{\Psi(Q) : Q \in \mathcal{K}_e^n\},$$

where $\Psi(Q) = \frac{1}{n\omega_n}\mathcal{E}(Q) - \frac{1}{p}\log\int_{S^{n-1}} \rho_Q^{-p} d\mu$. Then, there exists $c > 0$ such that $\mu = J_p(cK, \cdot)$.

We shall now adapt the above lemma to the discrete setting.

Suppose μ is an even discrete measure whose support is $\{\pm u_1, \pm u_2, \dots, \pm u_N\}$. Let $D_\mu \subset \mathcal{K}_o^n$ be the set of all origin-symmetric convex polytopes whose vertices are in the directions belonging to the set $\{\pm u_1, \pm u_2, \dots, \pm u_N\}$. It is obvious that if $K \in D_\mu$, then there exists $\rho_1, \dots, \rho_N > 0$ such that

$$D_\mu = \operatorname{conv}\{\pm\rho_1 u_1, \pm\rho_2 u_2, \dots, \pm\rho_N u_N\}.$$

The following lemma converts the even discrete L_p Aleksandrov problem into a maximization problem.

Lemma 3.2. *Suppose μ is an even discrete measure and $p \neq 0$. If there exists an origin-symmetric $K \in \mathcal{K}_e^n$ such that*

$$\Phi(K) = \sup_{Q \in D_\mu} \Phi(Q), \tag{3.1}$$

where $\Phi : \mathcal{K}_e^n \to \mathbb{R}$ *is given by*

$$\Phi(Q) = \frac{1}{n\omega_n}\mathcal{E}(Q) - \frac{1}{p}\log \sum_{u_i \in \operatorname{supp}\mu} \rho_Q(u_i)^{-p}\mu(\{u_i\}), \tag{3.2}$$

then there exists $c > 0$ *such that*

$$\mu = J_p(cK, \cdot).$$

Proof. It is obvious that

$$\sup_{Q \in D_\mu} \Phi(Q) \leq \sup_{Q \in \mathcal{K}_e^n} \Phi(Q). \tag{3.3}$$

On the other side, for each $Q \in \mathcal{K}_e^n$, let

$$\widetilde{Q} = \operatorname{conv}\{\rho_Q(u_i)u_i : u_i \in \operatorname{supp}\mu\}.$$

Then, $\rho_{\widetilde{Q}}(u_i) \geq \rho_Q(u_i)$ for each $u_i \in \operatorname{supp}\mu$ and $h_{\widetilde{Q}}(v) \leq h_Q(v)$ for each $v \in S^{n-1}$. This implies that

$$\Phi(Q) \leq \Phi(\widetilde{Q}).$$

This, in combination with (3.3), shows that

$$\Phi(K) = \sup_{Q \in D_\mu} \Phi(Q) = \sup_{Q \in \mathcal{K}_e^n} \Phi(Q).$$

Note that when the given measure μ is discrete, $\Psi(\cdot) = \Phi(\cdot)$. According to Lemma 3.1, there exists $c > 0$ such that $\mu = J_p(cK, \cdot)$. □

4. Solving the optimization problem

Lemma 4.1. *Suppose* μ *is an even discrete measure on* S^{n-1} *whose support is not contained in any great subspheres. Let* $Q^j \in D_\mu$ *be such that* $\max_{u \in S^{n-1}} \rho_{Q^j}(u) = 1$*. Assume there exists an origin-symmetric compact convex set* Q^0 *such that* Q^j *converges to* Q^0 *in Hausdorff metric. Then, by possibly taking a subsequence,*

$$\lim_{j \to \infty} \Phi(Q^j) = \Phi(Q^0).$$

Proof. Since $Q^j \in D_\mu$, by possibly taking a subsequence, we may assume $\rho_{Q^j}(u_{i_0}) = 1$ for some $u_{i_0} \in \operatorname{supp}\mu$. By the definition of support function and the fact that Q^j is origin-symmetric,

$$|u_{i_0} \cdot v| \leq h_{Q^j}(v) \leq 1, \qquad v \in S^{n-1}.$$

Hence,

$$|\log h_{Q^j}(v)| \leq -\log |u_{i_0} \cdot v|, \qquad v \in S^{n-1}.$$

Notice that $\log |u_{i_0} \cdot v|$ is an integrable function on S^{n-1}. Since Q^j converges to Q^0 in Hausdorff metric, h_{Q^j} converges to h_{Q^0} pointwise. This, combined with the fact that $h_{Q^0} > 0$ almost everywhere (since Q^0 has diameter bigger than 1), implies that $\log h_{Q^j}$ converges to $\log h_{Q^0}$ almost everywhere. By dominated convergence theorem,

$$\lim_{j \to \infty} \mathcal{E}(Q^j) = \mathcal{E}(Q^0). \tag{4.1}$$

On the other side, since Q^j converges to Q^0 in Hausdorff metric, we have

$$\lim_{j \to \infty} \rho_{Q^j}(u_i) = \rho_{Q^0}(u_i), \qquad \forall u_i \in \operatorname{supp}\mu.$$

Hence,

$$\lim_{j\to\infty} \sum_{u_i\in\operatorname{supp}\mu} \rho_{Q^j}(u_i)^{-p}\mu(\{u_i\}) = \sum_{u_i\in\operatorname{supp}\mu} \rho_{Q^0}(u_i)^{-p}\mu(\{u_i\}).$$

Since $\rho_{Q^j}(u_{i_0}) = 1$, we have $\sum_{u_i\in\operatorname{supp}\mu} \rho_{Q^j}(u_i)^{-p}\mu(\{u_i\}) > 0$. Hence,

$$\lim_{j\to\infty} \log \sum_{u_i\in\operatorname{supp}\mu} \rho_{Q^j}(u_i)^{-p}\mu(\{u_i\}) = \log \sum_{u_i\in\operatorname{supp}\mu} \rho_{Q^0}(u_i)^{-p}\mu(\{u_i\}). \tag{4.2}$$

Equations (4.1) and (4.2) imply that

$$\lim_{j\to\infty} \Phi(Q^j) = \Phi(Q^0).$$

□

Let S be a k-dimensional subspace of $\mathbb{R}^n$. Write $v \in S^{n-1}$ as

$$v = (v_2\cos\phi, v_1\sin\phi), \tag{4.3}$$

where $v_2 \in S^{k-1} \subset S$, $v_1 \in S^{n-k-1} \subset S^\perp$ and $0 \le \phi \le \pi/2$.

Lemma 4.2. *Suppose $u_1, \dots, u_N$ are N unit vectors such that they are not concentrated on any great subspheres. Let $f : S^{n-1} \to \mathbb{R}$ be defined as*

$$f(v) = \max_{s+1\le i\le N} |v \cdot u_i|,$$

where $1 \le s \le N-1$ is such that $S = \operatorname{span}\{u_1, \dots, u_s\}$ is a proper subspace of $\mathbb{R}^n$. Then, there exists constants $0 < c < 1$ and $0 < \delta_0 < \pi/2$ such that

$$f(v) \ge c,$$

for each $v = (v_2\cos\phi, v_1\sin\phi) \in S^{n-1}$ with $\phi > \pi/2 - \delta_0$. Here ϕ comes from the general polar coordinate expression (4.3)*.*

Proof. Note that f is uniformly continuous on S^{n-1}. Since $u_1, \dots, u_N$ are not concentrated on any great subspheres,

$$f(v) > 0, \qquad v \in S^\perp.$$

By continuity of f, there exists $0 < c_1 < 1$ such that

$$f(v) > c_1, \qquad v \in S^\perp. \tag{4.4}$$

Moreover, since f is uniformly continuous, there exists sufficiently small $0 < \delta_1 < 1$ such that $||v_1 - v_2|| < \delta_1$ implies $|f(v_1) - f(v_2)| < \frac{c_1}{2}$. This, when combined with (4.4), shows that there exists $0 < c < 1$ such that

$$f(v) \ge c, \qquad \text{for } v \in S^{n-1} \text{ with } \operatorname{dist}(v, S^\perp) < \delta_1.$$

The desired result now follows from the fact that we can find a sufficiently small $0 < \delta_0 < 1$ such that if $v \in S^{n-1}$ is such that $\phi > \frac{\pi}{2} - \delta_0$, then $\operatorname{dist}(v, S^\perp) < \delta_1$. □

The following lemma partitions S^{n-1} according to the support of a given measure μ.

Lemma 4.3. *Suppose $u_1, \dots, u_N$ are N unit vectors such that they are not concentrated on any great subsphere. Let $1 \le s \le N-1$ be such that $S = \operatorname{span}\{u_1, \dots, u_s\}$ is a proper subspace of $\mathbb{R}^n$. Let $R \ge 1$. For $1 \le \rho_1, \dots, \rho_s \le R$ and sufficiently* *small* $0 < t < 1$ *such that* $\arccos \frac{ct}{R} > \frac{\pi}{2} - \delta_0$ *where* c *and* δ_0 *come from Lemma 4.2, let*

$$K^t = conv\{\pm\rho_1 u_1, \ldots, \pm\rho_s u_s, \pm t u_{s+1}, \ldots, \pm t u_N\}.$$

Denote

$$\Omega_1 = \left\{ v \in S^{n-1} : \arccos \frac{ct}{R} < \phi < \frac{\pi}{2} \right\}$$
$$\Omega_2 = \left\{ v \in S^{n-1} : 0 \le \phi < \arccos \frac{t}{r_0} \right\}$$
$$\Omega_3 = \left\{ v \in S^{n-1} : \arccos \frac{t}{r_0} \le \phi \le \arccos \frac{ct}{R} \right\},$$

where $0 < r_0 < 1$ *is such that* $r_0 \le \max_{1 \le i \le s} |v \cdot u_i|$ *for all unit vectors* $v \in S$. *Then, for* $v \in \Omega_1$,

$$h_{K^t}(v) \le t, \qquad \text{and} \qquad h_{K^0}(v) \ge r_0 \cos\phi;$$

for $v \in \Omega_2$,

$$h_{K^t}(v) = h_{K^0}(v);$$

for $v \in \Omega_3$,

$$h_{K^t}(v) \le R, \qquad \text{and} \qquad h_{K^0}(v) \ge r_0 \cos\phi.$$

Proof. The existence of $0 < r_0 < 1$ such that $r_0 \le \max_{1 \le i \le s} |v \cdot u_i|$ for all unit vectors $v \in S$ follows from the fact that $u_1, \cdots, u_s$ spans S and that $\max_{1 \le i \le s} |v \cdot u_i|$ is a continuous function.

Throughout the proof, we will use the general polar coordinates (4.3).

Assume $v \in \Omega_1$. For $i = 1, \ldots, s$,

$$\rho_i |u_i \cdot v| = \rho_i \cos\phi |u_i \cdot v_2| \le R\cos\phi \le R \cdot \frac{ct}{R} \le t.$$

Since

$$h_{K^t}(v) = \max\left\{ \max_{i=1,\ldots,s} \rho_i |u_i \cdot v|, \max_{i=s+1,\ldots,N} t|u_i \cdot v| \right\},$$

we have $h_{K^t}(v) \le t$. On the other side, since $\rho_i \ge 1$, we have,

$$h_{K^0}(v) \ge \max_{1 \le i \le s} \rho_i \cos\phi |v_2 \cdot u_i| \ge r_0 \cos\phi.$$

Assume now, $v \in \Omega_2$. By the definition of support function,

$$h_{K^t}(v) \ge \max_{1 \le i \le s} \rho_i \cos\phi |v_2 \cdot u_i| \ge r_0 \cos\phi > t.$$

Since $t|v \cdot u| \le t$ for any $u \in S^{n-1}$, we have

$$h_{K^t}(v) = \max\left\{ \max_{i=1,\ldots,s} \rho_i |u_i \cdot v|, \max_{i=s+1,\ldots,N} t|u_i \cdot v| \right\} = \max_{i=1,\ldots,s} \rho_i |u_i \cdot v| = h_{K^0}(v).$$

Finally, let us assume $v \in \Omega_3$. By the fact that $\rho_{K^t} \le R$, it is apparent that $h_{K^t}(v) \le R$. The fact that $h_{K^0}(v) \ge r_0 \cos\phi$ follows from the same argument as in the case $v \in \Omega_1$. $\square$

The following lemma solves the optimization problem (3.1).

Lemma 4.4. *Let $-1 < p < 0$ and μ be an even discrete measure on S^{n-1} whose support is not contained in any great subspheres. Then, there exists $K \in \mathcal{K}_e^n$ such that*

$$\Phi(K) = \sup_{Q \in D_\mu} \Phi(Q),$$

where Φ is as defined in (3.2).

Proof. Suppose $\operatorname{supp} \mu = \{\pm u_1, \pm u_2, \dots, \pm u_N\}$ and

$$\mu = \sum_{i=1}^{N} \mu_i \left(\delta_{u_i} + \delta_{-u_i}\right).$$

Suppose $Q^j \subset D_\mu$ is a maximization sequence. Let

$$\rho_i^j = \rho_{Q^j}(u^i).$$

Since Φ is homogeneous of degree 0, we may rescale Q^j and assume $\max_i \rho_i^j = 1$. By Blaschke's selection theorem, after possibly taking a subsequence, we may assume that there exists an origin-symmetric compact convex set Q^0 such that Q^j converges to Q^0 in Hausdorff metric. Moreover,

$$\rho_i^0 := \rho_{Q^0}(u_i) = \lim_{j \to \infty} \rho_i^j.$$

By Lemma 4.1, after possibly taking another subsequence, we may assume

$$\Phi(Q^0) = \lim_{j \to \infty} \Phi(Q^j) = \sup_{Q \in D_\mu} \Phi(Q). \tag{4.5}$$

It remains to show that $o \in \operatorname{int} Q^0$.

We argue by contradiction and assume that there exists a proper subspace S of $\mathbb{R}^n$ such that $Q^0 \subset S$ and $\operatorname{span} Q^0 = S$. Let $k = \dim S$. Since S is a proper subspace, we have $1 \le k < n$. By relabelling, we may assume there exists $1 \le s < N$ such that

$$\pm u_1, \dots, \pm u_s \in S,$$

and

$$\pm u_{s+1}, \dots, \pm u_N \notin S.$$

Utilizing the fact that Φ is homogeneous of degree 0 again and that $\operatorname{span} Q^0 = S$, we may rescale Q^0 so that there exists $R \ge 1$ such that

$$1 \le \rho_1^0, \dots, \rho_s^0 \le R,$$

and

$$\rho_{s+1}^0, \dots, \rho_N^0 = 0.$$

For sufficiently small $0 < t < 1$ such that $\arccos \frac{ct}{R} > \frac{\pi}{2} - \delta_0$ where c and δ_0 come from Lemma 4.2, let

$$K^t = \operatorname{conv}\{\pm\rho_1 u_1, \dots, \pm\rho_s u_s, \pm t u_{s+1}, \dots, \pm t u_N\}.$$

Note that $K^t \in D_\mu$. We are going to reach the desired contradiction by showing that for some t, $\Phi(K^t) > \Phi(Q^0)$.

Towards this end, for each $K \in \mathcal{K}_e^n$, write

$$\mathcal{N}_p(K) = -\frac{1}{p}\log \sum_{u_i \in \operatorname{supp}\mu} \rho_K(u_i)^{-p}\mu_i = -\frac{1}{p}\log\left(2\sum_{i=1}^{N}\rho_K(u_i)^{-p}\mu_i\right).$$

Let $\Delta_1(t) = \frac{1}{n\omega_n}\mathcal{E}(K^t) - \frac{1}{n\omega_n}\mathcal{E}(Q^0)$ and $\Delta_2(t) = \mathcal{N}_p(K^t) - \mathcal{N}_p(Q^0)$. Note that

$$\Phi(K^t) - \Phi(Q^0) = \Delta_1(t) + \Delta_2(t). \tag{4.6}$$

By Lemma 4.3 and noticing that $\Omega_1, \Omega_2, \Omega_3$ is a partition of S^{n-1},

(4.7)

$$\begin{aligned}
n\omega_n\Delta_1(t) \geq & \left[-\int_{\Omega_1}\log t dv + \int_{\Omega_1}\log(r_0\cos\phi)dv\right] + \left[-\int_{\Omega_3}\log R dv + \int_{\Omega_3}\log(r_0\cos\phi)dv\right]\\
&= -k\omega_k(n-k)\omega_{n-k}\log t\int_{\arccos\frac{ct}{R}}^{\frac{\pi}{2}}\cos^{k-1}\phi\sin^{n-k-1}\phi d\phi\\
&\quad - k\omega_k(n-k)\omega_{n-k}\log R\int_{\arccos\frac{t}{r_0}}^{\arccos\frac{ct}{R}}\cos^{k-1}\phi\sin^{n-k-1}\phi d\phi\\
&\quad + k\omega_k(n-k)\omega_{n-k}\int_{\arccos\frac{t}{r_0}}^{\pi/2}\log(r_0\cos\phi)\cos^{k-1}\phi\sin^{n-k-1}\phi d\phi\\
&\geq -k\omega_k(n-k)\omega_{n-k}\log t\int_{\arccos\frac{ct}{R}}^{\frac{\pi}{2}}\cos^{k-1}\phi\sin^{n-k-1}\phi d\phi\\
&\quad - k\omega_k(n-k)\omega_{n-k}\log R\left(\arccos\frac{ct}{R} - \arccos\frac{t}{r_0}\right)\\
&\quad + k\omega_k(n-k)\omega_{n-k}\int_{\arccos\frac{t}{r_0}}^{\pi/2}\log(r_0\cos\phi)\cos^{k-1}\phi\sin^{n-k-1}\phi d\phi\\
&=: k\omega_k(n-k)\omega_{n-k}g_1(t).
\end{aligned}$$

Here, the constant r_0 comes from Lemma 4.3.

By the definition of K^t and $\Delta_2(t)$,

(4.8)

$$\begin{aligned}
\Delta_2(t) \geq & -\frac{1}{p}\log\left(2\sum_{i=1}^{s}\left(\rho_i^0\right)^{-p}\mu_i + 2\sum_{i=s+1}^{N}t^{-p}\mu_i\right) + \frac{1}{p}\log\left(2\sum_{i=1}^{s}\left(\rho_i^0\right)^{-p}\mu_i\right)\\
&= -\frac{1}{p}\log\frac{\sum_{i=1}^{s}\left(\rho_i^0\right)^{-p}\mu_i + \sum_{i=s+1}^{N}\mu_i t^{-p}}{\sum_{i=1}^{s}\left(\rho_i^0\right)^{-p}\mu_i}\\
&= -\frac{1}{p}\log\frac{a+bt^{-p}}{a}\\
&=: g_2(t),
\end{aligned}$$

where $a = \sum_{i=1}^{s}\left(\rho_i^0\right)^{-p}\mu_i > 0$ and $b = \sum_{i=s+1}^{N}\mu_i > 0$.

Note that $\lim_{t\to 0^+} g_1(t) = 0$. Indeed, for sufficiently small t,

(4.9)
$$\begin{aligned}
|g_1(t)| &\le |\log t|(\frac{\pi}{2} - \arccos\frac{ct}{R}) + \log R(\arccos\frac{ct}{R} - \arccos\frac{t}{r_0}) + \int_{\arccos\frac{t}{r_0}}^{\frac{\pi}{2}} |\log(r_0\cos\phi)|d\phi \\
&\le |\log t|\arcsin\frac{ct}{R} + \log R(\arcsin\frac{ct}{R} - \arcsin\frac{t}{r_0}) + \int_0^t \frac{1}{\sqrt{r_0^2-x^2}}|\log x|dx \\
&\le |\log t|\arcsin\frac{ct}{R} + \log R(\arcsin\frac{ct}{R} - \arcsin\frac{t}{r_0}) + \frac{2}{r_0}\int_0^t |\log x|dx \\
&\longrightarrow 0,
\end{aligned}$$

as $t \to 0$. Also, it is straightforward to see that $\lim_{t\to 0^+} g_2(t) = 0$.

Let $G(t) = \frac{k\omega_k(n-k)\omega_{n-k}}{n\omega_n} g_1(t) + g_2(t)$. From (4.7), (4.8), and (4.9), we see that

$$\Delta_1(t) + \Delta_2(t) \ge G(t), \tag{4.10}$$

and

$$\lim_{t\to 0^+} G(t) = 0. \tag{4.11}$$

By direct computation, for sufficiently small $t > 0$,

$$\begin{aligned}
g_1'(t) = &-\frac{1}{t}\int_{\arccos\frac{ct}{R}}^{\frac{\pi}{2}} \cos^{k-1}\phi\sin^{n-k-1}\phi d\phi \\
&- \log t\cos^{k-1}\left(\arccos\frac{ct}{R}\right)\sin^{n-k-1}\left(\arccos\frac{ct}{R}\right)\frac{1}{\sqrt{1-\left(\frac{ct}{R}\right)^2}}\frac{c}{R} \\
&- \log R\left(-\frac{1}{\sqrt{1-\left(\frac{ct}{R}\right)^2}}\frac{c}{R} + \frac{1}{\sqrt{r_0^2-t^2}}\right) \\
&+ \log\left(r_0\cos(\arccos\frac{t}{r_0})\right)\cos^{k-1}(\arccos\frac{t}{r_0})\sin^{n-k-1}(\arccos\frac{t}{r_0})\frac{1}{\sqrt{r_0^2-t^2}} \\
\ge &-\frac{\arcsin\frac{ct}{R}}{t} - \log R\left(-\frac{1}{\sqrt{1-\left(\frac{ct}{R}\right)^2}}\frac{c}{R} + \frac{1}{\sqrt{r_0^2-t^2}}\right) \\
&+ \log t\Bigg[-\cos^{k-1}\left(\arccos\frac{ct}{R}\right)\sin^{n-k-1}\left(\arccos\frac{ct}{R}\right)\frac{1}{\sqrt{1-\left(\frac{ct}{R}\right)^2}}\frac{c}{R} \\
&\qquad + \cos^{k-1}(\arccos\frac{t}{r_0})\sin^{n-k-1}(\arccos\frac{t}{r_0})\frac{1}{\sqrt{r_0^2-t^2}}\Bigg] \\
=: &\ C(t) + \log t\cdot D(t),
\end{aligned}$$

where $C(t)$ and $D(t)$ are bounded terms when $t > 0$ is sufficiently small.

On the other side, for $t > 0$ sufficiently small,

$$g_2'(t) = \frac{b}{a+t^{-p}b}t^{-p-1} \ge \frac{b}{2a}t^{-p-1}.$$

Hence,

$$G'(t) \geq \frac{k\omega_k(n-k)\omega_{n-k}}{n\omega_n}C(t) + \frac{k\omega_k(n-k)\omega_{n-k}}{n\omega_n}D(t)\log t + \frac{b}{2a}t^{-p-1}. \tag{4.12}$$

Since $-1 < p < 0$, when $t > 0$ is sufficiently small, the right side of (4.12) is positive. Hence, there exists $\delta_1 > 0$ such that $G'(t) > 0$ for each $t \in (0, \delta_1)$. This, combined with (4.11), implies that there exists $t_0 > 0$ such that $G(t_0) > 0$. By (4.10), this implies that $\Delta_1(t_0) + \Delta_2(t_0) > 0$. By (4.6), $\Phi(K^{t_0}) > \Phi(Q^0)$. But, this is a contradiction to (4.5). $\square$

Lemmas 3.2 and 4.4 immediate imply:

Theorem 4.5. *Suppose $p \in (-1, 0)$ and μ is a non-zero, even, discrete, finite, Borel measure on S^{n-1}. There exists an origin-symmetric polytope $K \in \mathcal{K}_e^n$ such that $\mu = J_p(K, \cdot)$ if and only if μ is not concentrated entirely on any great subspheres.*

Proof. The "only if" part is obvious while the "if" part follows from Lemmas 3.2 and 4.4. $\square$

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