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Smooth solutions to the L_p dual Minkowski problem

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Abstract In this paper, we consider the L_p dual Minkowski problem by geometric variational method. Using anisotropic Gauss-Kronecker curvature flows, we establish the existence of smooth solutions of the L_p dual Minkowski problem when $pq \geq 0$ and the given data is even. If $f \equiv 1$, we show under some restrictions on p and q that the only even, smooth, uniformly convex solution is the unit ball.

Keywords dual curvature measure · Minkowski problem · L_p dual Minkowski problem · Aleksandrov problem · dual Minkowski problem

Mathematics Subject Classification (2000) 52A38 · 35J20

1 Introduction

The classical Minkowski problem is a most fundamental problem in the study of convex bodies. It asks for necessary and sufficient conditions on a given measure so that it is the surface area measure of a convex body. The influence of the classical Minkowski problem reaches far beyond convex geometric

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analysis to fields such as PDE, differential geometry, and functional analysis. In particular, special cases of the Minkowski problem include the problem of prescribing Gauss curvature and a Monge-Ampère equation. One important application of the classical Minkowski problem is that it connects geometry with functional analysis. Using the solution of the classical Minkowski problem, Zhang [59] “translated” Petty’s affine projection inequality to a stronger affine version of the classical Sobolev inequality.

Since the classical Minkowski problem, many Minkowski type problems have been introduced and extensively studied, notably, the L_p Minkowski problem. The L_p Minkowski problem is the problem of prescribing L_p surface area measure which was introduced by Lutwak [41]. When the given measure has a density function, the L_p Minkowski problem reduces to an arguably much harder Monge-Ampère equation. In fact, two critical cases of the L_p Minkowski problem—the logarithmic Minkowski problem ($p = 0$) and the centro-affine Minkowski problem ($p = -n$)—still remain open. The L_p Minkowski problem when $p > 1$ was solved by Lutwak [41] when the given measure is even and by Chou & Wang [20] without evenness assumptions. Important contributions to critical cases of the L_p Minkowski problem include [11, 23, 42, 45, 62–64] and their references. The solution of L_p Minkowski problem was used to establish an L_p affine Sobolev inequality which is stronger than the L_p Sobolev inequality [27, 43, 44].

In a groundbreaking work [31], Lutwak, Yang, & Zhang (LYZ), and the second named author proposed a new family of geometric measures called dual curvature measures—“dual” to Federer’s curvature measures in the classical Brunn-Minkowski theory. Definitions of dual curvature measures are given in Section 2. In [31], the dual Minkowski problem—the problem of prescribing dual curvature measures—was posed. The dual Minkowski problem contains problems such as the Aleksandrov problem and the logarithmic Minkowski problem as special cases. The problem quickly became the center of attention leading to works such as [9, 28, 32, 35, 37, 60, 61]. The dual Minkowski problem, when $0 < q < n$ and the given measure is even, was first considered in [31]. A sufficient condition for the case when $0 < q < n$ is an integer was later given in [61]. The same condition was simultaneously shown to be necessary by [9]. Very recently, the sufficiency of the condition when $q > 0$ is a non-integer was given in [12]. The solutions there use variational method and elaborate integral estimates. In Li, Sheng, & Wang [37], the associated Monge-Ampère equation was thoroughly investigated. In particular, the solution when $q > 0$ and the given density function is even was given using geometric flow. It should be noted that the dual Minkowski problem when $q > 0$, especially the case when the given data in non-even, is still widely open.

A recent surprising result by LYZ [46] revealed a unifying family of geometric measures—the (p, q)th dual curvature measures—that contains all the aforementioned measures as special cases. This new family of geometric measures suggests that different theories—the classical Brunn-Minkowski theory, the L_p Brunn-Minkowski theory, and the dual Brunn-Minkowski theory—naturally arising in the journey of exploring measures, invariants, inequalities involving convex bodies could all turn out to be a part of the new L_p dual Brunn-Minkowski theory. The (p, q)th dual curvature measures are the fundamental geometric measures in the L_p dual Brunn-Minkowski theory, whose definitions and special cases are to follow in Section 2. The following L_p dual Minkowski problem was posed in [46]:

Problem 1.1 (The L_p dual Minkowski problems)**.** *Given a nonzero finite Borel measure μ on the unit sphere S^{n-1} and real numbers p, q, what are the necessary and sufficient conditions so that μ is equal to the (p, q)th dual curvature measure of some convex body K containing the origin in its interior?*

When the given measure μ has a density f, the L_p dual Minkowski problems becomes the following Monge-Ampère type equation on S^{n-1}:

$$\frac{h^{1-p}}{(h^2 + |\nabla h|^2)^{\frac{n-q}{2}}} \det(\nabla^2 h + hI) = f, \tag{1.1}$$

where f is a given positive smooth function on S^{n-1}, h is the unknown function, I is the identity matrix, and ∇h and $\nabla^2 h$ are the gradient and the Hessian of h on the unit sphere S^{n-1} with respect to an orthonormal basis respectively.

In [33], the existence and uniqueness of smooth solution of (1.1) were obtained by using the method of continuity for prescribed smooth function f and $p > q$. In this paper, equation (1.1) for the case when $pq \geq 0$ and f is even, will be thoroughly investigated.

Since the (p, q)th dual curvature measures contain all the aforementioned geometric measures, the L_p dual Minkowski problem, as well as (1.1), contains all the aforementioned Minkowski problems. When $p = 1$ and $q = n$, equation (1.1) corresponds to the classical Minkowski problem, see, e.g., [1, 2, 15–17, 21, 47–51, 56]. When $p = 0$ and $q = 0$, equation (1.1) corresponds to the Aleksandrov problem, the problem of prescribing Aleksandrov's integral curvature [3], which is the counterpart to the Minkowski problem. For $q = n$ and arbitrary p, equation (1.1) corresponds to the L_p Minkowski problem, see, e.g., [20, 29, 30, 34, 41, 42, 45, 57, 63, 64]. In particular, it contains the logarithmic Minkowski problem ($p = 0, q = n$), see for example [11], and the centro-affine Minkowski problem ($p = -n, q = n$), see for example [20, 63]. Both these problems are still unsolved. For $p = 0$ and an arbitrary q, equation (1.1) corresponds to the dual Minkowski problem.

Besides the fact that the existence part of many cases of L_p dual Minkowski problem being unsolved, results on the uniqueness part are even more scarce. Of all the aforementioned Minkowski problems, only the classical Minkowski problem and the Aleksandrov problem have their uniqueness problems completely settled. In fact, for the logarithmic Minkowski problem, uniqueness result was only found when the given measure is even in the planar case [10]. For the dual Minkowski problem, uniqueness result was only found when $q < 0$, see [60].

The following results will be established.

Theorem 1.2. *Suppose $pq \geq 0$. For any even, positive, smooth function f,*

1. *when* $p \neq q$*, equation* (1.1) *has an origin symmetric, smooth, uniformly convex solution* h*;*
2. *when* $p = q$*, an origin symmetric, smooth, uniformly convex solution exists for the normalized equation*

$$\frac{h^{1-p}}{(h^2+|\nabla h|^2)^{\frac{n-q}{2}}}\det(\nabla^2 h + hI) = \left(\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}\right) f.$$

The above theorem contains many interesting special cases. When $p = q = 0$, this is the even Aleksandrov problem. When $q = n$, this is the even L_p Minkowski problem for $p \geq 0$. In particular, when $p = 0$, this is the even logarithmic Minkowski problem. The case $p = 0$ in Theorem 1.2, which corresponds to the even dual Minkowski problem, was obtained by Li, Sheng, & Wang [37] using an anisotropic Gauss-Kronecker curvature flow. Inspired by their findings, we shall establish Theorem 1.2 using a different geometric flow.

The main idea is to find a suitable functional and a suitable anisotropic Gauss-Kronecker curvature flow that converges to equation (1.1) as $t \to \infty$. The difficulties are to obtain uniform positive lower and upper bounds of the support function and principal curvatures along the flows, which is precisely the reason that a new flow has to be adopted. In particular, we consider the following anisotropic Gauss-Kronecker curvature flow

$$\begin{cases} \dfrac{\partial X(x,t)}{\partial t} = -f(x)\rho^{n-q}h^p\mathcal{K}\dfrac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}\nu + X(x,t), \\ X(x,0) = X_0(x), \end{cases} \tag{1.2}$$

and a functional related with flow (1.2)

$$\Phi_{p,q}(K) = \log\|h\|_{f,p} - \log\|\rho\|_q, \quad \forall p,q \in \mathbb{R}. \tag{1.3}$$

Here we used the notation

$$\|g\|_{f,p} = \left(\frac{\int_{S^{n-1}} g^p(x)f(x)dx}{\int_{S^{n-1}} f(x)dx}\right)^{\frac{1}{p}},$$

where f, g are positive functions on S^{n-1} and $p \neq 0$. When $p = 0$, we write

$$\|g\|_{f,0} = \lim_{p\to 0}\|g\|_{f,p} = \exp\left(\frac{\int_{S^{n-1}}\log g(x)f(x)dx}{\int_{S^{n-1}} f(x)dx}\right)$$

In particular, when the density function $f \equiv 1$, we simply write $\|g\|_p$ for $\|g\|_{f,p}$.

The flow (1.2) is different from Li-Sheng-Wang's in [37] even in the case $p = 0$. The normalization factor $\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}$ makes sure that $\|\rho\|_q$ remains unchanged during the whole flow process, which is critical in obtaining uniform positive bounds.

Delicate estimates are needed to achieve uniform positive lower and upper bounds for the principal curvatures. In [37], Li, Sheng, & Wang studied the normalized contracting flow of polar bodies to establish these estimates. Here, we work directly on the original flow.

The use of geometric flow in Minkowski problems goes back to Firey [22] where he used the Gauss curvature flow as a model for the wearing of tumbling stones. Since then, the evolution of hypersurfaces by their Gauss curvature has been studied extensively, see for example, [4–8, 13, 14, 18, 19, 24, 37, 55, 58].

Partial uniqueness result will also be demonstrated.

Theorem 1.3. *Suppose $p \geq -n$, $q \leq \min\{n, n+p\}$, and $p \neq q$. Then h is an even, smooth, uniformly convex solution to* (1.1) *for $f \equiv 1$ if and only if $h \equiv 1$.*

Again, the above theorem contains the even L_p Minkowski problem when $0 < p < n$ and the even logarithmic Minkowski problem when the given measure has a constant density as special cases.

The paper is organized in the following way. After a short section of Preliminaries, the geometric flow and its associated functional will be introduced in Section 3. A priori estimates will be demonstrated in Sections 4 and 5. Theorem 1.2 will be carried out in Section 6 while the proof of Theorem 1.3 will be in Section 7. Last but not least, a duality relation for the equation (1.1) is to be demonstrated in Section 7, which may be of separate interest.

2 Preliminaries

2.1 Convex body and surface area measure

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. Denote by $\mathcal{K}_0^n$ the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and denote by $\mathcal{K}_e^n$ the class of origin-symmetric convex bodies. Let $K \in \mathcal{K}_0^n$. The radial function ρ_K is defined by

$$\rho_K(x) = \max\{\lambda : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

For $u \in S^{n-1}$, there is $\rho_K(u)u \in \partial K$.

The support function h_K of K is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The radial function and the support function are related,

$$\begin{aligned} h_K(v) &= \sup_{u \in S^{n-1}} \{\rho_K(u) u \cdot v\}, \\ \frac{1}{\rho_K(u)} &= \sup_{v \in S^{n-1}} \frac{u \cdot v}{h_K(v)}. \end{aligned}$$

For a convex body $K \in \mathcal{K}_0^n$, the polar body K^* of K is

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for all } \ y \in K\}.$$

Support and radial functions of the convex body and its polar are related in the following way,

$$h_K(x) = \rho_{K^*}^{-1}(x), \quad \rho_K(x) = h_{K^*}^{-1}(x). \tag{2.1}$$

For a convex body K and a Borel set $\omega \subset S^{n-1}$, the set $\nu_K^{-1}(\omega)$ is the inverse image of ω under the Gauss map ν_K of K. Associated with each convex body K is a Borel measure, S_K, on S^{n-1}, called the *surface area measure* of K and defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)) = \int_{\nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x),$$

for each Borel set $\omega \subseteq S^{n-1}$.

The L_p surface area measure $S_p(K, \cdot)$ of K is defined by Lutwak [41],

$$S_p(K, \omega) = \int_{\nu_K^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} \, d\mathcal{H}^{n-1}(x).$$

It follows that

$$S_p(K, \cdot) = h_K^{1-p} S_K.$$

2.2 L_p Dual curvature measures

Let $K \in \mathcal{K}_0^n$. Huang, LYZ [31] proposed a fundamental family of geometric measures in the dual Brunn-Minkowski theory—the dual curvature measure $\tilde{C}_q(K, \cdot)$ for each $q \in \mathbb{R}$. The measures were defined using what they call *local dual parallel bodies* and have the following integral representations:

$$\tilde{C}_q(K, \omega) = \frac{1}{n} \int_{\alpha_K^*(\omega)} \rho_K^q(u) \, du \quad \text{for each Borel set } \omega \subset S^{n-1},$$

where $\alpha_K^*(\omega)$ is the reverse radial Gauss image of ω given by

$$\alpha_K^*(\omega) = \{u \in S^{n-1} : u\rho_K(u) \in \nu_K^{-1}(\omega)\}.$$

Geometric measures in the classical Brunn-Minkowski theory, the L_p Brunn-Minkowski theory, and the dual Brunn-Minkowski theory had never been thought to be connected. But the recent paper by LYZ [46] suggests that they can be unified as special cases of a new family of geometric measures. For a convex body $K \in \mathcal{K}_0^n$ and real numbers p, q, define the measure $\tilde{C}_{p,q}(K, \cdot)$ on S^{n-1} by

$$\tilde{C}_{p,q}(K, \omega) = \frac{1}{n} \int_{\alpha_K^*(\omega)} h_K(\alpha_K(u))^{-p} \rho_K(u)^q \, du, \text{ for each Borel set } \omega \subset S^{n-1}.$$

Here $\alpha_K(u)$ is the radial Gauss map of K that takes $u \in S^{n-1}$ to the outer unit normal of K at the boundary point $\rho_K(u)u$. The map $\alpha_K(u)$ is defined almost everywhere on S^{n-1} with respect to the spherical Lebesgue measure.

Obviously, when $p = 0$, $\tilde{C}_{0,q}(K,\cdot)$ gives the q-th dual curvature measure. When $q = n$, $\tilde{C}_{0,n}$ is the cone-volume measure,

$$\tilde{C}_{0,n}(K,\cdot) = V_K = \frac{1}{n} h_K S_K,$$

and thus $\tilde{C}_{p,n}(K,\cdot)$ gives the L_p surface area measure,

$$\tilde{C}_{p,n}(K,\cdot) = h_K^{-p} V_K = \frac{1}{n} h_K^{1-p} S(K,\cdot) = \frac{1}{n} S_p(K,\cdot).$$

If K is a convex body that has C^2 boundary with positive curvature and contains the origin in its interior, then $\tilde{C}_{p,q}(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure,

$$\frac{d\tilde{C}_{p,q}(K,v)}{dv} = \frac{1}{n} h_K^{1-p} (h_K^2 + |\nabla h_K|^2)^{\frac{q-n}{2}} \det(\nabla^2 h_K + h_K I).$$

For simplicity, h and ρ will be used to replace h_K and ρ_K when it is clear what the underlying convex body is.

It is worthwhile to note that L_p dual curvature measures are valuations on the set of convex bodies containing the origin in their interiors. Valuations have been the objects of many recent works, see, for example, Haberl & Parapatits [25, 26], Ludwig [38, 39], Ludwig & Reitzner [40], Schuster [52, 53], Schuster & Wannerer [54] and the references therein.

3 Geometric flow and its associated functional

In this section, we shall introduce the geometric flow and its associated functional for solving the L_p dual Minkowski problem.

For readers' convenience, the associated Monge-Ampère type equation to the L_p dual Minkowski problem is restated here,

$$\frac{h^{1-p}}{(h^2 + |\nabla h|^2)^{\frac{n-q}{2}}} \det(\nabla^2 h + hI) = f(x), \quad x \in S^{n-1}. \tag{3.1}$$

Recall the normalized equation

$$\frac{h^{1-p}}{(h^2 + |\nabla h|^2)^{\frac{n-q}{2}}} \det(\nabla^2 h + hI) = f(x) \frac{\int_{S^{n-1}} \rho^q du}{\int_{S^{n-1}} h^p f dx}. \tag{3.2}$$

Note that when $p \neq q$, by homogeneity, if (3.2) has a convex solution h, then $\left[\frac{\int_{S^{n-1}} \rho^q du}{\int_{S^{n-1}} h^p f dx}\right]^{-\frac{1}{q-p}} h$ is a solution of (3.1). Hence, a solution of (3.2) will immediately lead to a solution of (3.1) in the case $p \neq q$.

Let K_0 be a smooth, closed, origin symmetric, and uniformly convex hypersurface in $\mathbb{R}^n$ enclosing the origin in its interior. Consider the following anisotropic Gauss-Kronecker curvature flow

$$\begin{cases} \dfrac{\partial X(x,t)}{\partial t} = -f(x)\rho^{n-q}h^p\mathcal{K}\dfrac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}\nu + X(x,t), \\ X(x,0) = X_0(x), \end{cases} \tag{3.3}$$

where $\mathcal{K} = \det(\nabla^2 h + hI)^{-1}$ is the Gauss-Kronecker curvature of hypersurface ∂K_t parametrized by $X(\cdot,t) : S^{n-1} \to \mathbb{R}^n$. Here $\nu = x$ is the outer unit normal at $X(x,t)$, $\rho(u,t)$ is the radial function of K_t, and $h(x,t)$ is the support function of K_t. We show that the functional $\Phi_{p,q}(K)$ (see (1.3)) is non-increasing along the flow (3.3).

By the definition of support function, we know $h(x,t) = x \cdot X(x,t)$. Hence,

$$\begin{cases} \dfrac{\partial h(x,t)}{\partial t} = -f(x)\rho^{n-q}h^p\mathcal{K}\dfrac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx} + h(x,t), \\ h(x,0) = h_0(x). \end{cases} \tag{3.4}$$

Note

$$\begin{aligned} \frac{1}{\rho(u,t)}\frac{\partial \rho(u,t)}{\partial t} =& \frac{1}{h(x,t)}[\nabla h \cdot x_t + h_t] - \frac{u \cdot x_t}{u \cdot x} \\ =& \frac{1}{h(x,t)}\frac{\partial h(x,t)}{\partial t} + \frac{1}{h(x,t)}[(\nabla h - \rho(u,t)u) \cdot x_t] \\ =& \frac{1}{h(x,t)}\frac{\partial h(x,t)}{\partial t}. \end{aligned}$$

Hence,

$$\begin{cases} \dfrac{\partial \rho(u,t)}{\partial t} = -f(x)\rho^{n-q+1}h^{p-1}\mathcal{K}\dfrac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx} + \rho(u,t), \\ \rho(u,0) = \rho_0(u). \end{cases} \tag{3.5}$$

Recall that

$$\Phi_{p,q}(K_t) = \log \|h(\cdot,t)\|_{f,p} - \log \|\rho(\cdot,t)\|_q.$$

The following lemma shows that the functional $\Phi_{p,q}$ is non-increasing along the flow (3.3).

Lemma 3.1. *For any $p, q \in \mathbb{R}$, the functional $\Phi_{p,q}$ is non-increasing along the flow* (3.3)*. In particular,*

$$\frac{d\Phi_{p,q}(K_t)}{dt} \leq 0,$$

and the equality holds if and only if K_t satisfies the elliptic equation (3.2)*.*

Proof. We first consider the case where $p, q \neq 0$.

From (3.4) and (3.5), we have

$$\begin{aligned}
\frac{d\Phi_{p,q}(K_t)}{dt} =& \frac{1}{\int_{S^{n-1}} h^p f dx} \int_{\mathbb{S}^{n-1}} h(x,t)^{p-1} \frac{\partial h(x,t)}{\partial t} f(x) dx \\
&- \frac{1}{\int_{S^{n-1}} \rho^q du} \int_{\mathbb{S}^{n-1}} \rho(u,t)^{q-1} \frac{\partial \rho(u,t)}{\partial t} du \\
=& \int_{\mathbb{S}^{n-1}} \frac{\partial h(x,t)}{\partial t} \left[\frac{h(x,t)^{p-1} f(x)}{\int_{S^{n-1}} h^p f dx} - \frac{\rho(u,t)^{q-1}}{\int_{S^{n-1}} \rho^q du} \frac{\rho(u,t)}{h(x,t)} \left| \frac{\partial u}{\partial x} \right| \right] dx \\
=& \int_{\mathbb{S}^{n-1}} \frac{\partial h}{\partial t} \left[\frac{h^{p-1} f}{\int_{S^{n-1}} h^p f dx} - \frac{1}{\int_{S^{n-1}} \rho^q du} \frac{\rho^q}{h} \frac{h}{\rho^n \mathcal{K}} \right] dx \\
=& - \int_{\mathbb{S}^{n-1}} \frac{[-f \rho^{n-q} h^p \mathcal{K} \frac{\int_{S^{n-1}} \rho^q du}{\int_{S^{n-1}} h^p f dx} + h]^2}{h \rho^{n-q} \mathcal{K} \int_{S^{n-1}} \rho^q du} dx \qquad (3.6) \\
\leqslant& 0,
\end{aligned}$$

and equality holds if and only if

$$f(x) \rho^{n-q} h^p \mathcal{K} \frac{\int_{S^{n-1}} \rho^q du}{\int_{S^{n-1}} h^p f dx} = h(x,t),$$

which implies (3.2). The lemma follows similarly when either $p = 0$ or $q = 0$. □

Lemma 3.1 suggests that in order to obtain a solution of (3.2), one only needs to show the flow (3.3) is uniformly bounded in C^2.

4 C^0, C^1 bounds

The C^0 and C^1 bounds for the flow (3.2) will be established in this section.

We first note that the normalization factor $\frac{\int_{S^{n-1}} \rho^q du}{\int_{S^{n-1}} h^p f dx}$ guarantees that the norm $\|\rho\|_q$ remains unchanged throughout the flow.

Lemma 4.1. *Suppose $q \in \mathbb{R}$. Let f be an even, positive, smooth function on S^{n-1}, and K_t be an origin symmetric, uniformly convex solution to the flow (3.3). Then*

$$\|\rho(\cdot, t)\|_q = const,$$

for $t \geq 0$. Here $\rho(\cdot, t)$ is the radial function of K_t.

Proof. First, we consider the case when $q \neq 0$. By (3.5) and the fact that $\frac{h}{\mathcal{K}}dx = \rho^n du$, we have

$$\begin{aligned}\frac{d}{dt}\int_{S^{n-1}} \rho(u,t)^q du =& q\int_{S^{n-1}} \rho^{q-1}\frac{d\rho}{dt}du\\ =& q\int_{S^{n-1}} \rho^{q-1}\left(-f\rho^{n-q+1}h^{p-1}\mathcal{K}\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}} h^p f dx} + \rho\right)du\\ =& q\left(-\int_{S^{n-1}} h^{p-1}f\rho^n\mathcal{K}du\cdot\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}} h^p f dx} + \int_{S^{n-1}}\rho^q du\right)\\ \equiv& 0.\end{aligned}$$

The desired result when $q \neq 0$ follows immediately from the definition of $\|\rho(\cdot,t)\|_q$.

The case when $q = 0$ follows in a similar way. By (3.5) and the fact that $\frac{h}{\mathcal{K}}dx = \rho^n du$,

$$\begin{aligned}\frac{d}{dt}\int_{S^{n-1}} \log\rho(u,t)du =& \int_{S^{n-1}} \rho^{-1}\frac{d\rho}{dt}du\\ =& \int_{S^{n-1}}\left(-f\rho^n h^{p-1}\mathcal{K}\frac{\int_{S^{n-1}} du}{\int_{S^{n-1}} h^p f dx} + 1\right)du\\ =& -\int_{S^{n-1}} h^{p-1}f\rho^n\mathcal{K}du\cdot\frac{\int_{S^{n-1}} du}{\int_{S^{n-1}} h^p f dx} + \int_{S^{n-1}} du\\ \equiv& 0.\end{aligned}$$

This, combined with the definition of $\|\rho(\cdot,t)\|_0$, immediately completes the proof. □

The following lemma establishes the C^0 bounds.

Lemma 4.2. *Suppose $pq \geq 0$. Let f be an even, positive, smooth function on S^{n-1}, and K_t be an origin symmetric, uniformly convex solution to the flow* (3.3)*. Then there is a positive constant C independent of t such that*

$$\frac{1}{C} \leqslant h(x,t) \leqslant C, \quad \forall\ (x,t) \in S^{n-1}\times(0,+\infty), \tag{4.1}$$

$$\frac{1}{C} \leqslant \rho(u,t) \leqslant C, \quad \forall\ (u,t) \in S^{n-1}\times(0,+\infty), \tag{4.2}$$

Here $h(x,t)$ and $\rho(u,t)$ are the support function and the radial function of K_t, respectively.

Proof. Let $\rho_{\max}(t) = \max_{S^{n-1}}\rho(u,t) = \rho(u_1(t),t)$ for some $u_1(t) \in S^{n-1}$ and $\rho_{\min}(t) = \min_{S^{n-1}}\rho(u,t) = \rho(u_2(t),t)$ for some $u_2(t) \in S^{n-1}$. Note that (4.1) and (4.2) are equivalent. Hence, for upper bound (or lower bound) we only need to establish (4.1) or (4.2). For simplicity, for $u \in S^{n-1}$, let $g_u(x) = |x\cdot u|$ be a function on $\mathbb{R}^n$.

The proof is separated into two cases: 1. $q > 0$ (which implies $p \geq 0$); 2. $q \leq 0$.

Let us first consider the case $q > 0$.

Since K_t is origin symmetric, by the definition of support function,

$$h(x,t) \geq \rho_{\max}|x \cdot u_1| = \rho_{\max} g_{u_1}(x).$$

Hence, by Lemma 3.1 and Lemma 4.1, for $p \geq 0$,

$$\|h_0\|_{f,p} \geq \|h(\cdot,t)\|_{f,p} \geq \|\rho_{\max} g_{u_1}\|_{f,p} = \rho_{\max}\|g_{u_1(t)}\|_{f,p}. \tag{4.3}$$

For $p > 0$, since f is positive and continuous, we have

$$\|g_u\|_{f,p} \geq c_0\|g_u\|_p \geq c_0 c_1, \tag{4.4}$$

where $c_0 > 0$ is the lower bound of f and the last inequality is because $\|g_u\|_p$ is invariant of the choice of u. Combining (4.3) and (4.4) gives us the upper bound of (4.2) when $p > 0$.

For $p = 0$, since f is positive and continuous,

$$\|f\|_1 \log \|g_u\|_{f,0} \geq n\omega_n \log c_0 + \int_{S^{n-1}} \log|x \cdot u| dx \geq c_2, \tag{4.5}$$

where c_2 is a constant (not necessarily positive) independent of u since the integral $\int_{S^{n-1}} \log|x \cdot u| dx$ does not depend on u. Hence $\|g_u\|_{f,0}$ is uniformly positively bounded from below. This, combined with (4.3), shows the upper bound of (4.2) when $p > 0$.

For the lower bound, we prove it by contradiction. Assume that $h(x,t)$ is not uniformly bounded away from 0. Thus, K_t converges to a convex body contained in a lower-dimensional subspace. This means that

$$\rho(u,t) \to 0 \tag{4.6}$$

as $t \to \infty$ almost everywhere with respect to the spherical Lebesgue measure. Combined with bounded convergence theorem, (4.6) implies that when $q > 0$, we have $\|\rho(\cdot,t)\|_q \to 0$, which is a contradiction to Lemma 4.1.

This completes the proof for the case when $q > 0$. For the rest of the proof, we consider the case $q \leq 0$.

By Lemma 4.1 and (2.1), we have

$$\|h^*(\cdot,t)\|_{-q} = \text{const},$$

where we used $h^*(\cdot,t)$ for the support function of K_t^*, the polar body of K_t. Note that $-q$ is a nonnegative number. Using a similar argument in (4.3), (4.4),and (4.5), we conclude that $\rho^*(\cdot,t)$, the radial function of K_t^* is uniformly bounded from above. Using (2.1) again allows us to conclude the lower bound in (4.1).

For the upper bound, note that by Lemma 3.1 and Lemma 4.1,

$$\|h_0\|_{f,p} \geq \|h(\cdot,t)\|_{f,p}. \tag{4.7}$$

When $p < 0$, this implies that

$$\int_{S^{n-1}} h(x,t)^p f(x)dx$$

is uniformly bounded from below. By (2.1), this implies that

$$\int_{S^{n-1}} \rho^*(x,t)^{-p} f(x)dx$$

is bounded from below. A similar proof-by-contradiction argument as before shows that $\rho^*(x,t)$ is uniformly bounded from below, which implies that $h(x,t)$ is uniformly bounded from above.

When $p \geq 0$, (4.7) implies that $\|h(\cdot,t)\|_{f,p}$ is bounded from above. By (4.3) and (4.5), we conclude that $\rho(\cdot,t)$ is uniformly bounded from above.

This completes the proof for the case when $q \leq 0$. □

C^1 bound immediately follows.

Corollary 4.3. *Let f be an even, positive, smooth function on S^{n-1}, and $K_t \in C^{k+2}$ be an origin symmetric, uniformly convex solution to the flow* (3.3). *Then there is a positive constant C independent of t such that*

$$|\nabla h(x,t)| \leqslant C, \quad \forall\ (x,t) \in S^{n-1} \times (0,+\infty),$$

and

$$|\nabla \rho(u,t)| \leqslant C, \quad \forall\ (u,t) \in S^{n-1} \times (0,+\infty).$$

Proof. The desired results immediate follows from Lemma 4.2 and the identities

$$h = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla\rho|^2}}, \quad \rho^2 = h^2 + |\nabla h|^2.$$

□

5 C^2 bound

In this section, we establish the lower and upper bounds of principal curvatures. This shows that (3.4) and (3.5) are uniformly parabolic.

Theorem 5.1. *Let f be an even, positive, smooth function on S^{n-1}, and K_t be an origin symmetric, smooth, uniformly convex solution to the flow* (3.3). *Then there is a positive constant C independent of t such that the principal curvatures $\kappa_1, \kappa_2, \cdots, \kappa_{n-1}$ of K_t satisfy*

$$\frac{1}{C} \leqslant \kappa_i(x,t) \leqslant C, \quad \forall\ (x,t) \in S^{n-1} \times (0,+\infty).$$

Proof. The proof is divided into two parts: in the first part, we derive an upper bound for the Gauss curvature $\mathcal{K}(x,t)$; in the second part, we derive an upper bound for the principal radii $b_{ij} = h_{ij} + h\delta_{ij}$.

⧫ Step 1: Prove $\mathcal{K} \leqslant C$.

Consider the auxiliary function

$$Q(x,t) = \frac{f(x)\rho^{n-q}h^p\mathcal{K}\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx} - h}{h-\varepsilon_0} \equiv \frac{-h_t}{h-\varepsilon_0},$$

where

$$\varepsilon_0 = \frac{1}{2}\min_{S^{n-1}\times(0,+\infty)} h(x,t) > 0.$$

For any fixed $t \in (0,+\infty)$, suppose $\max_{S^{n-1}} Q(x,t)$ is attained at x_0. Then, at x_0,

$$0 = \nabla_i Q = \frac{-h_{ti}}{h-\varepsilon_0} + \frac{h_t h_i}{(h-\varepsilon_0)^2},$$

and

$$\begin{aligned}
0 \geqslant \nabla_{ij} Q =& \frac{-h_{tij}}{h-\varepsilon_0} + \frac{h_{ti}h_j + h_{tj}h_i + h_t h_{ij}}{(h-\varepsilon_0)^2} - 2\frac{h_t h_i h_j}{(h-\varepsilon_0)^3}\\
=& \frac{-h_{tij}}{h-\varepsilon_0} + \frac{h_t h_{ij}}{(h-\varepsilon_0)^2}.
\end{aligned}$$

Hence

$$-h_{tij} - h_t\delta_{ij} \leqslant -\frac{h_t h_{ij}}{h-\varepsilon_0} - h_t\delta_{ij} = \frac{-h_t}{h-\varepsilon_0}[h_{ij} + (h-\varepsilon_0)\delta_{ij}] = Q[b_{ij} - \varepsilon_0\delta_{ij}].$$

At x_0, we also have

$$\begin{aligned}
\partial_t Q =& \frac{-h_{tt}}{h-\varepsilon_0} + \frac{h_t^2}{(h-\varepsilon_0)^2}\\
=& \frac{f}{h-\varepsilon_0}\left[\frac{\partial(\rho^{n-q}h^p\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx})}{\partial t}\mathcal{K} + \rho^{n-q}h^p\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}\frac{\partial[\det(\nabla^2 h + hI)]^{-1}}{\partial t}\right]\\
&+ Q + Q^2,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial(\rho^{n-q}h^p)}{\partial t} =& (n-q)\rho^{n-q-1}\frac{\partial\rho(u(x,t),t)}{\partial t}h^p + \rho^{n-q}ph^{p-1}\frac{\partial h(x,t)}{\partial t}\\
=& (n-q)\rho^{n-q-2}[hh_t + \sum h_k h_{kt}]h^p - \rho^{n-q}ph^{p-1}(h-\varepsilon_0)Q\\
=& (n-q)\rho^{n-q-2}h^p Q[\varepsilon_0 h - \rho^2] - \rho^{n-q}ph^{p-1}(h-\varepsilon_0)Q,
\end{aligned}$$

$$\begin{aligned}\frac{\partial\left[\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}\right]}{\partial t} &= -\left[\frac{\int_{S^{n-1}}\rho^q du}{(\int_{S^{n-1}}h^p f dx)^2}\right]\frac{\partial\int_{S^{n-1}}h^p f dx}{\partial t}\\ &= -\left[\frac{\int_{S^{n-1}}\rho^q du}{(\int_{S^{n-1}}h^p f dx)^2}\right]\int_{S^{n-1}} ph^{p-1} f\frac{\partial h}{\partial t}dx\\ &= \left[\frac{\int_{S^{n-1}}\rho^q du}{(\int_{S^{n-1}}h^p f dx)^2}\right]\int_{S^{n-1}} ph^{p-1} f(h-\varepsilon_0)Q dx\\ &\le \left[\frac{\int_{S^{n-1}}\rho^q du}{(\int_{S^{n-1}}h^p f dx)^2}\right]\int_{S^{n-1}} |p| h^p f dx\cdot Q(x_0,t),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial[\det(\nabla^2 h+hI)]^{-1}}{\partial t} &= -[\det(\nabla^2 h+hI)]^{-2}\frac{\partial[\det(\nabla^2 h+hI)]}{\partial b_{ij}}[h_{tij}+h_t\delta_{ij}]\\ &\leqslant [\det(\nabla^2 h+hI)]^{-2}\frac{\partial[\det(\nabla_{ii} h+hI)]}{\partial b_{ij}}Q[b_{ij}-\varepsilon_0\delta_{ij}]\\ &\leqslant \mathcal{K}Q[(n-1)-\varepsilon_0(n-1)\mathcal{K}^{\frac{1}{n-1}}].\end{aligned}$$

So we have at x_0

$$\begin{aligned}\partial_t Q \le & \frac{1}{h-\varepsilon_0}\left\{CQ^2 + f\rho^{n-q}h^p\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}\mathcal{K}Q\left[(n-1)-\varepsilon_0(n-1)\mathcal{K}^{\frac{1}{n-1}}\right]\right\}\\ &+Q+Q^2.\end{aligned}$$

If $Q \gg 1$,

$$\frac{1}{C_0}\mathcal{K}\leqslant Q\leqslant C_0\mathcal{K},$$

which implies

$$\partial_t Q\leqslant C_1 Q^2(C_2-\varepsilon_0 Q^{\frac{1}{n-1}})<0.$$

Hence

$$Q(x_0,t)\leqslant C,$$

and for any (x,t)

$$\mathcal{K}(x,t)=\frac{(h-\varepsilon_0)Q(x,t)+h}{f(x)\rho^{n-q}h^p\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}}\le\frac{(h-\varepsilon_0)Q(x_0,t)+h}{f(x)\rho^{n-q}h^p\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}}\leqslant C.$$

♦ Step 2: Prove $\kappa_i \geqslant \frac{1}{C}$.
Consider the auxiliary function

$$w(x,t)=\log\lambda_{\max}(\{b_{ij}\})-A\log h+B|\nabla h|^2,$$

where $A = 2B \max\limits_{S^{n-1}\times(0,+\infty)} |\nabla h|^2 + 1$, and

$$B = |n-q| \frac{\max\limits_{S^{n-1}\times(0,+\infty)} \rho^2 + 4 \max\limits_{S^{n-1}\times(0,+\infty)} |\nabla h|^2}{\min\limits_{S^{n-1}\times(0,+\infty)} \rho^4} + 1$$

are positive constants, and $\lambda_{\max}(\{b_{ij}\})$ denotes the maximal eigenvalue of $\{b_{ij}\}$. For convenience, we write $\{b^{ij}\}$ for $\{b_{ij}\}^{-1}$.

For any fixed $t \in (0,+\infty)$, assume $\max\limits_{S^{n-1}} w(x,t)$ is attained at some point $x_0 \in S^{n-1}$. By a rotation, we may assume

$$\{b_{ij}(x_0,t)\} \ \text{ is diagonal, and } \lambda_{\max}(\{b_{ij}\})(x_0,t) = b_{11}(x_0,t).$$

Hence, to show $\kappa_i \geqslant \frac{1}{C}$, it suffices to prove that $b_{11} \leqslant C$.

The function

$$\widetilde{w}(x,t) = \log b_{11} - A\log h + B|\nabla h|^2$$

attains local maximum at x_0. Thus, we have at x_0,

$$\begin{aligned} 0 = \nabla_i \widetilde{w} =& b^{11}\nabla_i b_{11} - A\frac{h_i}{h} + 2B\sum h_k h_{ki} \\ =& b^{11}[h_{i11} + h_1\delta_{i1}] - A\frac{h_i}{h} + 2Bh_i h_{ii}, \end{aligned} \tag{5.1}$$

and

$$0 \geqslant \nabla_{ii}\widetilde{w} = b^{11}\nabla_{ii}b_{11} - (b^{11})^2(\nabla_i b_{11})^2 - A\left[\frac{h_{ii}}{h} - \frac{h_i^2}{h^2}\right] + 2B\left[\sum h_k h_{kii} + h_{ii}^2\right].$$

At x_0, we also have

$$\begin{aligned} \partial_t \widetilde{w} =& b^{11}\partial_t b_{11} - A\frac{h_t}{h} + 2B\sum h_k h_{kt} \\ =& b^{11}[h_{11t} + h_t] - A\frac{h_t}{h} + 2B\sum h_k h_{kt}. \end{aligned}$$

From the equation (3.4), we know

$$\log(h - h_t) = -\log\det(\nabla^2 h + hI) + \log\left[f\rho^{n-q}h^p \frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}} h^p f dx}\right]. \tag{5.2}$$

Let

$$\phi(x,t) = \log\left[f\rho^{n-q}h^p \frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}} h^p f dx}\right].$$

Differentiating (5.2) gives

$$\begin{aligned} \frac{h_k - h_{kt}}{h - h_t} =& -\sum b^{ij}\nabla_k b_{ij} + \nabla_k \phi \\ =& -\sum b^{ii}[h_{kii} + h_i\delta_{ik}] + \nabla_k\phi, \end{aligned}$$

and

$$\frac{h_{11}-h_{11t}}{h-h_t}-\frac{(h_1-h_{1t})^2}{(h-h_t)^2}=-\sum b^{ii}\nabla_{11}b_{ii}+\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2+\nabla_{11}\phi.$$

By the Ricci identity, we have

$$\nabla_{11}b_{ii}=\nabla_{ii}b_{11}-b_{11}+b_{ii}.$$

Then we have

$$\begin{aligned}
\frac{\partial_t\widetilde{w}}{h-h_t}=&b^{11}\left[\frac{h_{11t}-h_{11}}{h-h_t}+\frac{h_{11}+h-h+h_t}{h-h_t}\right]-A\frac{1}{h}\frac{h_t-h+h}{h-h_t}+2B\frac{\sum h_kh_{kt}}{h-h_t}\\
=&b^{11}\left[-\frac{(h_1-h_{1t})^2}{(h-h_t)^2}+\sum b^{ii}\nabla_{11}b_{ii}-\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2-\nabla_{11}\phi\right]\\
&+\frac{1-A}{h-h_t}-b^{11}+\frac{A}{h}+2B\frac{\sum h_kh_{kt}}{h-h_t}\\
\leqslant&b^{11}\left[\sum b^{ii}(\nabla_{ii}b_{11}-b_{11}+b_{ii})-\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2\right]\\
&-b^{11}\nabla_{11}\phi+\frac{1-A}{h-h_t}+\frac{A}{h}+2B\frac{\sum h_kh_{kt}}{h-h_t}\\
\leqslant&\sum b^{ii}\left[(b^{11})^2(\nabla_i b_{11})^2+A\left(\frac{h_{ii}}{h}-\frac{h_i^2}{h^2}\right)-2B(\sum h_kh_{kii}+h_{ii}^2)\right]\\
&-b^{11}\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2-b^{11}\nabla_{11}\phi+\frac{1-A}{h-h_t}+\frac{A}{h}+2B\frac{\sum h_kh_{kt}}{h-h_t}\\
\leqslant&\sum b^{ii}\left[A\left(\frac{h_{ii}+h-h}{h}-\frac{h_i^2}{h^2}\right)\right]+2B\sum h_k\left[-\sum b^{ii}h_{kii}+\frac{h_{kt}}{h-h_t}\right]\\
&-2B\sum b^{ii}[b_{ii}-h]^2-b^{11}\nabla_{11}\phi+\frac{1-A}{h-h_t}+\frac{A}{h}\\
\leqslant&\sum b^{ii}\left[A\left(\frac{b_{ii}}{h}-1\right)\right]+2B\sum h_k\left[\frac{h_k}{h-h_t}+b^{kk}h_k-\nabla_k\phi\right]\\
&-2B\sum b^{ii}[b_{ii}^2-2b_{ii}h]-b^{11}\nabla_{11}\phi+\frac{1-A}{h-h_t}+\frac{A}{h}\\
\leqslant&-2B\sum h_k\nabla_k\phi-b^{11}\nabla_{11}\phi+(2B|\nabla h|-A)\sum b^{ii}\\
&-2B\sum b_{ii}+4B(n-1)h+\frac{2B|\nabla h|^2+1-A}{h-h_t}+\frac{nA}{h}.
\end{aligned}$$

From $\phi(x,t) = \log\left[f\rho^{n-q}h^p \frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}\right]$ and (5.1), we know

$$\begin{aligned}
&-2B\sum h_k\nabla_k\phi - b^{11}\nabla_{11}\phi\\
=&-2B\sum h_k\left[\frac{f_k}{f}+(n-q)\frac{hh_k+h_kh_{kk}}{\rho^2}+p\frac{h_k}{h}\right]\\
&-b^{11}\left[\frac{ff_{11}-f_1^2}{f^2}+(n-q)\frac{hh_{11}+h_1^2+h_{11}^2+\sum h_kh_{k11}}{\rho^2}\right]\\
&-b^{11}\left[-2(n-q)\frac{(hh_1+h_1h_{11})^2}{\rho^4}+p\frac{hh_{11}-h_1^2}{h^2}\right]\\
\\
\leqslant&C_1B+(n-q)\sum\frac{h_k}{\rho^2}[-2Bh_kh_{kk}-b^{11}h_{k11}]+C_2b^{11}\\
&+b^{11}\left[|n-q|\frac{h|h_{11}|+h_{11}^2}{\rho^2}+4|n-q|\frac{h_1^2h_{11}^2}{\rho^4}+|p|\frac{h|h_{11}|}{h^2}\right]\\
\leqslant&C_1B+(n-q)\sum\frac{h_k}{\rho^2}\left[b^{11}h_1\delta_{k1}-A\frac{h_k}{h}\right]+C_2b^{11}\\
&+b^{11}\left[|n-q|\frac{h(b_{11}-h)+(b_{11}-h)^2}{\rho^2}\right.\\
&\left.+4|n-q|\frac{h_1^2(b_{11}-h)^2}{\rho^4}+|p|\frac{h(b_{11}-h)}{h^2}\right]\\
\leqslant&C_1B+C_3b^{11}+C_4+|n-q|\frac{\rho^2+4h_1^2}{\rho^4}b_{11}.
\end{aligned}$$

It follows that

$$\frac{\partial_t\widetilde{w}}{h-h_t}\leqslant C_1B+C_3b^{11}+C_4-B\sum b_{ii}+4B(n-1)h+\frac{nA}{h}<0,$$

provided $b_{11}\gg 1$. Hence,

$$\widetilde{w}(x_0,t)\leqslant C.$$

As a result,

$$w(x_0,t)=\widetilde{w}(x_0,t)\leqslant C,$$

This tells us the principal radii is bounded from above, or equivalently $\kappa_i \geqslant \frac{1}{C}$. □

6 Existence of solutions to the flow

From the uniform estimates of the support function and the principal curvatures, we obtain the existence of solutions to the flow (3.3).

Theorem 6.1. *Suppose $pq \geq 0$. Let K_0 be a smooth, closed, origin symmetric, uniformly convex body in $\mathbb{R}^n$ enclosing the origin in its interior, and f be an even, positive, smooth function on S^{n-1}. Then the flow* (3.3) *has a smooth solution K_t for all time $t > 0$. Moreover, there exists an origin-symmetric, smooth, uniformly convex solution K of* (3.2).

Proof. By the a priori estimates in Theorem 4.2 and Theorem 5.1, equation (3.3) is uniformly parabolic. Estimates for higher derivatives follows from the standard regularity theory of uniformly parabolic equations Krylov [36]. Hence we obtain the long time existence and regularity of solutions for the flow (3.3). Moreover, we have

$$||h||_{C^{i,j}_{x,t}(S^{n-1}\times[0,\infty))} \tag{6.1}$$

is bounded for each pairs of nonnegative integers i and j.

By Lemma 3.1,

$$\frac{d\Phi_{p,q}(K_t)}{dt} \leq 0.$$

If there exists a time $t_0 > 0$ such that

$$\left.\frac{d\Phi_{p,q}(K_t)}{dt}\right|_{t=t_0} = 0,$$

we simply set $K = K_{t_0}$ and we are done by the equality condition in Lemma 3.1. Hence, from here on, we assume

$$\frac{d\Phi_{p,q}(K_t)}{dt} < 0. \tag{6.2}$$

By Arzelà-Ascoli theorem and a diagonal argument, we may extract a subsequence $\{t_s\} \subset (0,\infty)$ such that there exists a smooth function $h(x)$ and

$$||h(x,t_s) - h(x)||_{C^i(S^{n-1})} \to 0, \tag{6.3}$$

for each nonnegative integer i as $s \to \infty$.

Since $h(x,t_s)$ converges uniformly to $h(x)$, the function $h(x)$ is a support function. Let K be the convex body determined by h. By (6.3), K is origin symmetric and uniformly convex.

Since (6.1) is bounded for each pairs of i and j, the function $\frac{d^2\Phi_{p,q}(K_t)}{dt^2}$ is bounded. This implies that $\frac{d\Phi_{p,q}(K_t)}{dt}$ is uniformly continuous. Therefore, $\Phi_{p,q}(K_t)$ is a bounded function in t whose first-order derivative is negative and uniformly continuous. An application of Fundamental Theorem of Calculus together with the uniform continuity of $\frac{d\Phi_{p,q}(K_t)}{dt}$ shows,

$$\lim_{t\to\infty} \frac{d\Phi_{p,q}(K_t)}{dt} = 0. \tag{6.4}$$

By the a priori estimates obtained in Sections 4 and 5, there exists $\delta_0 > 0$ such that

$$\begin{aligned}\frac{d\Phi_{p,q}(K_t)}{dt}\Big|_{t=t_s} =& -\int_{\mathbb{S}^{n-1}} \frac{[-f\rho^{n-q}h^p\mathcal{K}\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}+h]^2}{h\rho^{n-q}\mathcal{K}\int_{S^{n-1}}\rho^q du}dx\Big|_{t=t_s}\\ \leqslant& -\delta_0\int_{S^{n-1}}(f\rho^{n-q}h^p\mathcal{K}\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}-h)^2 dx\Big|_{t=t_s}.\end{aligned}$$

Here, the first equality is due to (3.6). Let $s \to \infty$. We have

$$0 = \lim_{s\to\infty}\frac{d\Phi_{p,q}(K_t)}{dt}\Big|_{t=t_s} \leqslant -\delta_0\int_{S^{n-1}}(f\rho^{n-q}h^p\mathcal{K}\frac{\int_{S^{n-1}}\rho^q du}{\int_{S^{n-1}}h^p f dx}-h)^2 dx \leqslant 0.$$

Here h and ρ are support function and radial function of the limit convex body K. This allows us to conclude that h satisfies (3.2).

□

7 A duality relation and the uniqueness of the solution to the L_p-dual Minkowski problem

The following duality relation for the Monge-Ampère equation (1.1) might be of separate interest.

Theorem 7.1. *If the support function h of a smooth, strictly convex body K satisfies equation* (1.1)*, then the support function h^* of the polar body K^* satisfies the following equation*

$$\frac{{h^*}^{1+q}}{({h^*}^2+|\nabla h^*|^2)^{\frac{n+p}{2}}}\det(\nabla^2 h^*+h^*I)=\frac{1}{f}, \tag{7.1}$$

where the right side is evaluated at each $x \in S^{n-1}$ and the left side is evaluated at the corresponding normal vector $u = \nu_{K^*}(x\rho_{K^*}(x))$.

Proof. Since K^* is the polar body of K, we have $h^* = \frac{1}{\rho}$, $h = \frac{1}{\rho^*}$, $|\nabla h^*|^2 + {h^*}^2 = {\rho^*}^2$, and $h^2 + |\nabla h|^2 = \rho^2$. Moreover, by direct computation,

$$\begin{aligned}\mathcal{K} =& \frac{1}{\det(\nabla^2 h+hI)} = \frac{h^{1-p}}{f(h^2+|\nabla h|^2)^{\frac{n-q}{2}}}\\ =& \frac{\left(|\nabla h^*|^2+{h^*}^2\right)^{\frac{p-1}{2}}}{f{h^*}^{q-n}},\end{aligned}$$

and

$$\begin{aligned}\mathcal{K} =& (|\nabla\rho|^2+\rho^2)^{-\frac{n+1}{2}}\rho^{-2n+4}\det(-\rho\rho_{ij}+2\rho_i\rho_j+\rho^2\delta_{ij})\\ =& \left(|\nabla h^*|^2+{h^*}^2\right)^{-\frac{n+1}{2}}{h^*}^{n+1}\det(\nabla^2 h^*+h^*I).\end{aligned}$$

Combining these equations leads to (7.1). □

When $f \equiv 1$, the uniqueness of the solution to (3.1) can be obtained. The idea comes from Firey [22]. See also [37].

Theorem 7.2. *Suppose $p \geq -n$, $q \leq \min\{n, n+p\}$, and $p \neq q$. Then h is an even, smooth, uniformly convex solution to the equation*

$$\frac{h^{1-p}}{(h^2+|\nabla h|^2)^{\frac{n-q}{2}}} \det(\nabla^2 h + hI) = 1 \tag{7.2}$$

if and only if $h \equiv 1$.

Proof. From equation (7.2), we get for $q \leq n$,

$$h \det(\nabla^2 h + hI) = h^p (h^2 + |\nabla h|^2)^{\frac{n-q}{2}} \geq h^{n+p-q}.$$

Using Theorem 7.1 and $p \geq -n$, the support function of the polar body of our solution, h^*, satisfies

$$h^* \det(\nabla^2 h^* + h^* I) = h^{*-q} (h^{*2} + |\nabla h^*|^2)^{\frac{n+p}{2}} \geq h^{*n+p-q}.$$

Using the volume differential forms $h \det(\nabla^2 h + hI)$ and $h^* \det(\nabla^2 h^* + h^* I)$, we have

$$\begin{aligned}
\mathrm{Vol}(K)\mathrm{Vol}(K^*) =& \frac{1}{n} \int_{S^{n-1}} h \det(\nabla^2 h + hI) dx \cdot \frac{1}{n} \int_{S^{n-1}} h^* \det(\nabla^2 h^* + h^* I) dx \\
\geq& \frac{1}{n} \int_{S^{n-1}} h^{n+p-q} dx \cdot \frac{1}{n} \int_{S^{n-1}} h^{*n+p-q} dx \\
\geq& \frac{1}{n^2} \left[\int_{S^{n-1}} (hh^*)^{\frac{n+p-q}{2}} dx \right]^2 .
\end{aligned} \tag{7.3}$$

On the other hand, from the polar dual identity (2.1),

$$h(x)h^*(x) = \frac{h(x)}{\rho(x)} = \frac{\sup_{u \in S^{n-1}} \{\rho(u) u \cdot x\}}{\rho(x)} \geq 1.$$

It then follows from (7.3) and $q \leq n+p$ that

$$\mathrm{Vol}(K)\mathrm{Vol}(K^*) \geq \mathrm{Vol}(B_1)^2.$$

On the other hand, the Blaschke-Santaló inequality tells us

$$\mathrm{Vol}(K)\mathrm{Vol}(K^*) \leq \mathrm{Vol}(B_1)^2.$$

Thus, the equality in the Blaschke-Santaló inequality must hold, which implies that K must be an ellipsoid. Hence,

$$h^{n+1} \det(\nabla^2 h + hI) = c_0,$$

for some constant $c_0 > 0$. By substituting this into (7.2), we obtain

$$h^{p+n} (h^2 + |\nabla h|^2)^{\frac{n-q}{2}} = \frac{1}{c}.$$

This combined with the fact that the gradient of h at the maximum and minimum points are 0 imply that $\max h = \min h = c^{\frac{1}{q-p}}$, or K is a ball of radius $c^{\frac{1}{q-p}}$. Since $p \neq q$, plugging this into (7.2) quickly shows that $c = 1$.

$\square$

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